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STELLAR MOVEMENTS AND THE STRUCTURE OF THE UNIVERSE

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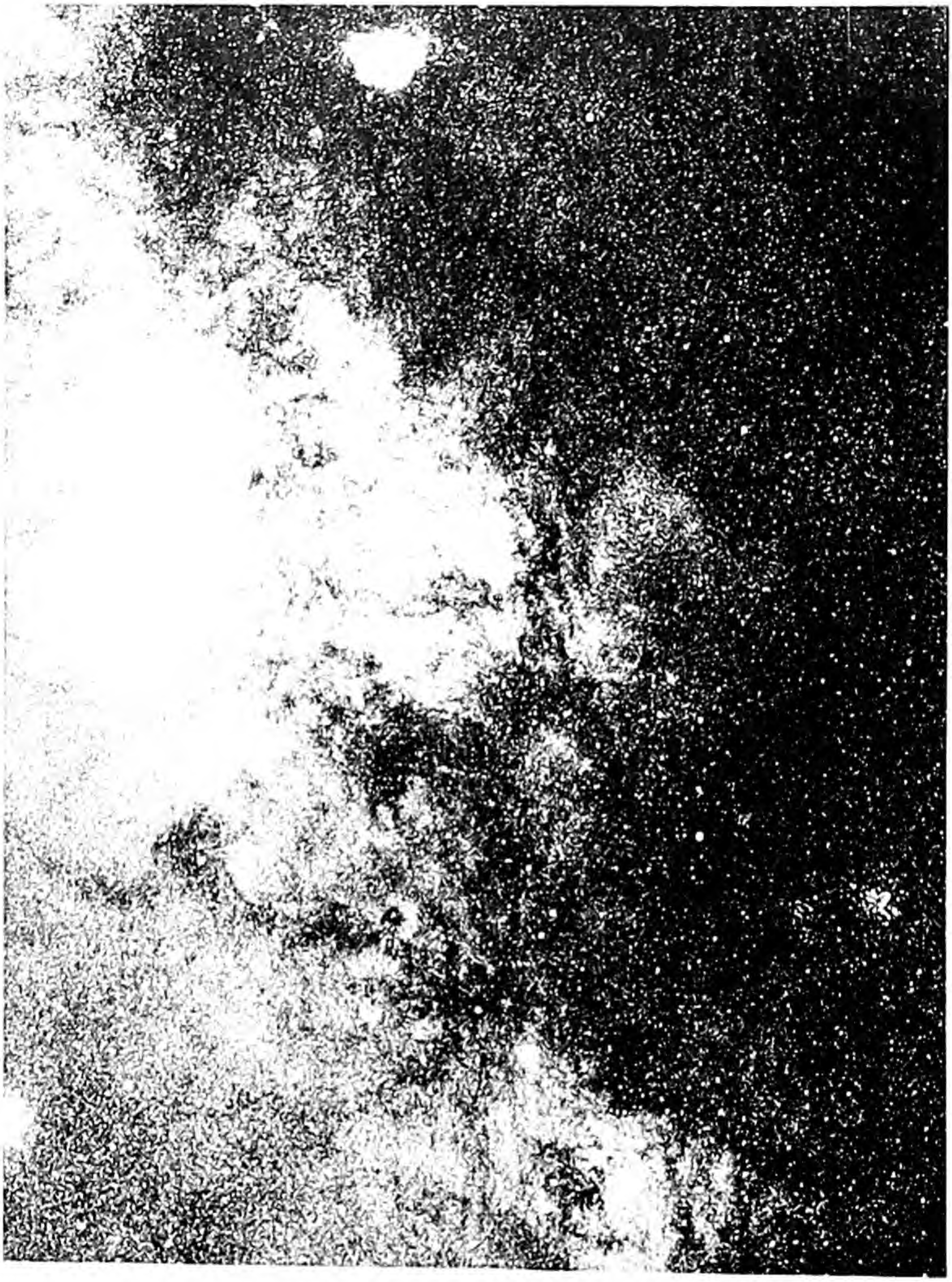

THE GREAT STAR-CLOUD IN SAGITTARIUS

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STELLAR MOVEMENTS AND THE STRUCTURE OF THE UNIVERSE

BY

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PREFACE

THE purpose of this monograph is to give an account of the present state of our knowledge of the structure of the sidereal universe. This branch of astronomy has become especially prominent during the last ten years; and many new facts have recently been brought to light. There is every reason to hope that the next few years will be equally fruitful; and it may seem hazardous at the present stage to attempt a general discussion of our knowledge. Yet perhaps at a time like the present, when investigations are being actively prosecuted, a survey of the advance made may be especially helpful.

The knowledge that progress will inevitably lead to a readjustment of ideas must instil a writer with caution; but I believe that excessive caution is not to be desired. There can be no harm in building hypotheses, and weaving explanations which seem best fitted to our present partial knowledge. These are not idle speculations if they help us, even temporarily, to grasp the relations of scattered facts, and to organise our knowledge.

No attempt has been made to treat the subject historically. I have preferred to describe the results of inves-

tigations founded on the most recent data rather than early pioneer researches. One inevitable result I particularly regret; many of the workers who have prepared the way for recent progress receive but scanty mention. Sir W. Herschel, Kobold, Seeliger, Newcomb and others would have figured far more prominently in a historical account. But it was outside my purpose to describe the steps by which knowledge has advanced; it is the present situation that is here surveyed.

So far as practicable I have endeavoured to write for the general scientific reader. It was impossible, without too great a sacrifice, to avoid mathematical arguments altogether; but the greater part of the mathematical analysis has been segregated into two chapters (VII and X). Its occasional intrusion into the remaining chapters will, it is hoped, not interfere with the readable character of the book.

I am indebted to Prof. G. E. Hale for permission to reproduce the two photographs of nebulæ (Plate 4), taken by Mr. G. W. Ritchey at the Mount Wilson Observatory, and to the Astronomer Royal, Dr. F. W. Dyson, for the three remaining plates taken from the Franklin-Adams Chart of the Sky. This represents but a small part of my obligation to Dr. Dyson; at one time and another nearly all the subjects treated in this book have been discussed between us, and I make no attempt to discriminate the ideas which I owe to him. There are many other astronomers from whose conversation, consciously or unconsciously, I have drawn material for this work.

I have to thank Mr. P. J. Melotte of the Royal Observatory, Greenwich, who kindly prepared the three Franklin-Adams photographs for reproduction.

Dr. S. Chapman, Chief Assistant at the Royal Observatory, Greenwich, has kindly read the proof-sheets, and I am grateful to him for his careful scrutiny and advice.

I also desire to record my great indebtedness to the Editor of this series of monographs, Prof. R. A. Gregory, for many valuable suggestions and for his assistance in passing this work through the press.

A. S. EDDINGTON.

THE OBSERVATORY, CAMBRIDGE.
April, 1914.

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CHAPTER I

THE DATA OF OBSERVATION

It is estimated that the number of stars which could be revealed by the most powerful telescopes now in use amounts to some hundreds of millions. One of the principal aims of stellar astronomy is to ascertain the relations and associations which exist among this multitude of individuals, and to study the nature and organisation of the great system which they constitute. This study is as yet in its infancy; and, when we consider the magnitude of the problem, we shall scarcely expect that progress towards a full understanding of the nature of the sidereal universe will be rapid. But active research in this branch of astronomy, especially during the last ten years, has led to many results, which appear to be well established. It has become possible to form an idea of the general distribution of the stars through space, and the general character of their motions. Though gaps remain in our knowledge, and some of the most vital questions are as yet without an answer, investigation along many different lines has elicited striking facts that may well be set down. It is our task in the following pages to coordinate these results, and review the advance that has been made.

Until recent years the study of the bodies of the solar system formed by far the largest division of astronomy; but with that branch of the subject we have nothing to do here. From our point of view the whole solar system is only a unit among myriads of similar units. The system of the stars is on a scale a million-fold greater than that of the planets; and stellar distances exceed a million times the distances of the comparatively well-known bodies which circulate round the Sun. Further, although we have taken the stars for our subject, not all branches of stellar astronomy fall within the scope of this book. It is to the relations between the stars—to the stars as a society—that attention is here directed. We are not concerned with the individual peculiarities of stars, except in so far as they assist in a broad classification according to brightness, stage of development, and other properties. Accordingly, it is not proposed to enter here into the more detailed study of the physical characters of stars; and the many interesting phenomena of Variable Stars and Novæ, of binary systems, of stellar chemistry and temperatures are foreign to our present aim.

The principal astronomical observations, on which the whole superstructure of fact or hypothesis must be based, may be briefly enumerated. The data about a star which are useful for our investigations are:—

1. Apparent position in the sky.
2. Magnitude.
3. Type of spectrum, or Colour.
4. Parallax.
5. Proper motion.
6. Radial velocity.

In addition, it is possible in some rather rare cases to find the mass or density of a star. This is a matter of importance, for presumably one star can only influence another by means of its gravitational attraction, which depends on the masses.

This nearly exhausts the list of characteristics which are useful in investigating the general problems of stellar distribution.* It is only in the rarest cases that the complete knowledge of a star, indicated under the foregoing six heads, is obtainable; and the indirect nature of most of the processes of investigation that are adopted is due to the necessity of making as much use as possible of the very partial knowledge that we do possess.

The observations enumerated already will now be considered in order. The apparent position in the sky needs no comment; it can always be stated with all the accuracy requisite.

Magnitude.—The magnitude of a star is a measure of its *apparent* brightness; unless the distance is known, we are not in a position to calculate the intrinsic or absolute brightness. Magnitudes of stars are measured on a logarithmic scale. Starting from a sixth magnitude star, which represents an arbitrary standard of brightness of traditional origin, but now fixed with sufficient precision, a star of magnitude 5 is one from which we receive 2·512 times more light. Similarly, each step of one magnitude downwards or upwards represents an increase or decrease of light in the ratio 1 : 2·512.† The number is so chosen that a difference of five magnitudes corresponds to a light ratio of $100 = (2{\cdot}512)^5$. The general formula is

$$\log_{10} \frac{L_1}{L_2} = -0{\cdot}4\,(m_1 - m_2),$$

where

L_1, L_2 are the intensities of the light from two stars
m_1, m_2 are their magnitudes.

Magnitude classifications are of two kinds: photometric (or visual), and photographic; for it is often found that of

* We should perhaps add that the separations and periods of binary stars are also likely to prove useful data.

† It is necessary to warn the reader that there are magnitude systems still in common use—that of the Bonn Durchmusterung, that used by many double-star observers, and even the magnitudes of Boss's Preliminary General Catalogue (1910)—which do not conform to this scale.

two stars the one which appears the brighter to the eye leaves a fainter image on a photographic plate. Neither of the two systems has been defined very rigorously; for, when stars are of different colours, a certain amount of personality exists in judging equality of light by the eye, and, if a photographic plate is used, differences may arise depending on the colour-sensitiveness of the particular kind of plate, or on the chromatic correction of the telescope object-glass. As the accuracy of magnitude-determinations improves, it will probably become necessary to adopt more precise definitions of the visual and photographic scales; but at present there appears to be no serious want of uniformity from this cause. But the distinction between the photometric and photographic magnitudes is very important, and the differences are large. The bluer the colour of a star, the greater is its relative effect on the photographic plate. A blue star and a red star of the same visual brightness may differ photographically by as much as two magnitudes.

The use of a logarithmic scale for measuring brightness possesses many advantages; but it is liable to give a misleading impression of the real significance of the numbers thus employed. It is not always realised how very rough are the usual measures of stellar brightness. If the magnitude of an individual star is not more than $0^{m}\cdot1$ in error, we are generally well satisfied; yet this means an error of nearly 10 per cent. in the light-intensity. Interpreted in that way it seems a rather poor result. An important part of stellar investigation is concerned with counts of the number of stars within certain limits of magnitude. As the number of stars increases about three-fold for each successive step of one magnitude, it is clear that all such work will depend very vitally on the absence of systematic error in the adopted scale of magnitudes; an error of two- or three-tenths of a magnitude would affect the figures profoundly. The establish-

ment of an accurate magnitude-system, with sequences of standard stars, has been a matter of great difficulty, and it is not certain that even now a sufficiently definitive system has been reached. The stars which come under notice range over more than twenty magnitudes, corresponding to a light ratio of 100,000,000 to 1. To sub-divide such a range without serious cumulative error would be a task of great difficulty in any kind of physical measurement.

The extensive researches of the Harvard Observatory, covering both hemispheres of the sky, are the main basis of modern standard magnitudes. The Harvard sequence of standard stars in the neighbourhood of the North Pole, extending by convenient steps as far as magnitude 21, at the limit reached with the 60-inch reflector of the Mount Wilson Observatory, now provides a suitable scale from which differential measures can be made. The absolute scale of the Harvard sequence of photographic magnitudes has been independently tested by F. H. Seares, at Mount Wilson, and S. Chapman and P. J. Melotte, at Greenwich. Both agree that from the tenth to the fifteenth magnitudes the scale is sensibly correct. But according to Seares a correction is needed between the second and ninth magnitudes, $1^{m}\cdot00$ on the Harvard scale being equivalent to $1^{m}\cdot07$ absolute. If this result is correct, the error introduced into statistical discussions must be quite appreciable.

For statistical purposes there are now available determinations of the magnitudes of the stars in bulk made at Harvard, Potsdam, Göttingen, Greenwich and Yerkes Observatories. The revised Harvard photometry gives the visual magnitudes of all stars down to about $6^{m}\cdot5$; the Potsdam magnitudes, also visual, carry us in a more limited part of the sky as far as magnitude $7^{m}\cdot5$. The Göttingen determinations, which are absolute determinations, independent of but agreeing very fairly with the Harvard

sequences, provide photographic magnitudes over a large zone of the sky for stars brighter than $7^m\cdot5$. The Yerkes investigation gives visual and photographic magnitudes of the stars within 17° of the North Pole down to $7^m\cdot5$. A series of investigations at Greenwich, based on the Harvard sequences, provides statistics for the fainter stars extending as far as magnitude 17, and is a specially valuable source for the study of the remoter parts of the stellar system.

So far we have been considering the apparent brightness of stars and not their intrinsic brightness. The latter quantity can be calculated when the distance of the star is known. We shall measure the absolute luminosity in terms of the Sun as unit. The brightness of the Sun has been measured in stellar magnitudes and may be taken to be $-26^m\cdot1$, that is to say it is 26·1 magnitudes brighter than a star of zero magnitude.* From this the luminosity L of a star, the magnitude of which is m, and parallax is ϖ'', is given by

$$\log_{10} L = 0\cdot2 - 0\cdot4 \times m - 2 \log_{10} \varpi.$$

The absolute magnitudes of stars differ nearly as widely as their apparent magnitudes. The feeblest star known is the companion to Groombridge 34, which is eight magnitudes fainter than the Sun. Estimates of the luminosities of the brightest stars are usually very uncertain; but, to take only results which have been definitely ascertained, the Cepheid Variables are *on an average* seven magnitudes brighter than the Sun. Probably this luminosity is exceeded by many of the Orion type stars. There is thus a range of at least fifteen magnitudes in the intrinsic brightness, or a light ratio of 1,000,000 to 1.

* The most recent researches give a value $-26\cdot5$ (Ceraski, *Annals of the Observatory of Moscow*, 1911). It is best, however, to regard the unit of luminosity as a conventional unit, roughly representing the Sun, and defined by the formula, rather than to keep changing the measures of stellar luminosity every time a better determination of the Sun's stellar magnitude is made.

Type of Spectrum.—For the type of spectrum various systems of classification have been used by astro-physicists, but that of the Draper Catalogue of Harvard Observatory is the most extensively employed in work on stellar distribution. This is largely due to the very complete classification of the brighter stars that has been made on this system. The classes in the supposed order of evolution are denoted by the letters :—

O, B, A, F, G, K, M, N.

A continuous gradation is recognised from O to M, and a more minute sub-division is obtained by supposing the transition from one class to the next to be divided into tenths. Thus B5A, usually abbreviated to B5, indicates a type midway between B and A ; G2 denotes a type between G and K, but more closely allied to the former. It is usual to class as Type A all stars from A0 to A9 ; but presumably it would be preferable to group together the stars from B6 to A5, and this principle has occasionally been adopted.

For our purposes it is not generally necessary to consider what physical peculiarities in the stars are represented by these letters, as the knowledge is not necessary for discussing the relations of motion and distribution. All that we require is a means of dividing the stars according to the stage of evolution they have attained, and of grouping the stars with certain common characteristics. It may, however, be of interest to describe briefly the principles which govern the classification, and to indicate the leading types of spectrum.

Tracing an imaginary star, as it passes through the successive stages of evolution from the earliest to the latest, the changes in its spectrum are supposed to pursue the following course. At first the spectrum consists wholly of diffuse bright bands on a faint continuous back-ground. The bands become fewer and narrower, and faint

absorption lines make their appearance; the first lines seen are those of the various helium series, the well-known Balmer hydrogen series, and the "additional" or "sharp" hydrogen series.* The last is a spectrum which had been recognised in the stars by E. C. Pickering in 1896 but was first produced artificially by A. Fowler in 1913. The bright bands now disappear, and in the remaining stages the spectrum consists wholly of absorption lines and bands, except in abnormal individual stars, which exceptionally show bright lines. The next phase is an enormous increase in the intensity of the true hydrogen spectrum, the lines becoming very wide and diffused; the other lines disappear. The lines H and K of calcium and other solar lines next become evident and gain in intensity. After this the hydrogen lines decline, though long remaining the leading feature of the spectrum; first a stage is reached in which the calcium spectrum becomes very intense, and afterwards all the multitudinous lines of the solar spectrum are seen. After passing the stage reached by our Sun, the chief feature is a shortening of the spectrum from the ultraviolet end, a further fading of the hydrogen lines, an increase in the number of fine absorption lines, and finally the appearance of bands due to metallic compounds, particularly the flutings of titanium oxide. The whole spectrum ultimately approximates towards that of sunspots.

Guided by these principles, we distinguish eight leading types, between which, however, there is a continuous series of gradations.

TYPE O (WOLF-RAYET TYPE)—The spectrum consists of bright bands on a faint continuous background; of these the most conspicuous have their centres at λλ 4633, 4651, 4686, 5693, and 5813. The type is divided into

* The practical work of A. Fowler and the theoretical researches of N. Bohr leave little doubt that this spectrum is due to helium, notwithstanding its simple numerical relation to the hydrogen spectrum.

five divisions, Oa, Ob, Oc, Od, and Oe, marked by varying intensities and widths of the bands. Further, in Od and Oe dark lines, chiefly belonging to the helium and helium-hydrogen series, make their appearance.

TYPE B (ORION TYPE)—This is often called the helium type owing to the prominence of the lines of that element. In addition there are some characteristic lines the origin of which is unknown, as well as both the "sharp" and Balmer series. The bright bands seen in Type O are no longer present; in fact, they disappear as early as the sub-division Oc5B, which is therefore usually reckoned the starting point of the Orion type.

TYPE A (SIRIAN TYPE).—The Balmer series of hydrogen is present in great intensity, and is far the most conspicuous feature of the spectrum. Other lines are present, but they are relatively faint.

TYPE F (CALCIUM TYPE).—The hydrogen series is still very conspicuous, but not so strong as in the preceding type. The narrow H and K lines of calcium have become very prominent, and characterise this spectrum.

TYPE G (SOLAR TYPE).—The Sun may be regarded as a typical star of this class, the numerous metallic lines having made their appearance.

TYPE K.—The spectrum is somewhat similar to the last. It is mainly distinguished by the fact that the hydrogen lines, which are still fairly strong in the G stars, are now fainter than some of the metallic lines.

TYPE M.—The spectrum is now marked by the appearance of flutings, due to titanium oxide. It is remarkable that the spectrum should be dominated so completely by this one substance. Two successive stages are recognised, indicated by the sub-divisions Ma and Mb. The long-period variable stars, which show bright hydrogen lines in addition to the ordinary characteristics of Type M, form the class Md.

TYPE N. — The regular progression of the types

terminates with Mb. There is no transition to Type N, and the relation of this to the foregoing types is uncertain. It is marked by characteristic flutings attributed to compounds of carbon. The stars of both the M and N types are of a strongly reddish tinge.

It is sometimes convenient to use the rather less detailed classification of A. Secchi. Strictly speaking his system relates to the visual spectrum, and the Draper notation to the photographic; but the two can well be harmonised.

Secchi's	Type	I.	includes	Draper	B and A
,,	,,	II.	,,	,,	F, G, and K
,,	,,	III.	,,	,,	M
,,	,,	IV.	,,	,,	N

As comparatively few of the stars in any catalogue belong to the last two types, this classification is practically a separation into two groups, which are of about equal size. This is a very useful division when the material is too scanty to admit of a more extended discussion.

From time to time there are indications that the Draper classification has not succeeded in separating the stars into really homogeneous groups. According to Sir Norman Lockyer, there are stars of ascending temperature and of descending temperature in practically every group; so that, for example, the stars enumerated under K are a mixture of two classes, one in a very early, the other in a late, stage of evolution. In the case of Type B, H. Ludendorff[1] has found considerable systematic differences in the measured radial motions of the stars classed by Lockyer as ascending and descending respectively (pointing, however, to real differences not of motion but of physical state, which have introduced an error into the spectroscopic measurements). E. Hertzsprung[2] has pointed out that the presence or absence of the *c* character on Miss Maury's classification (*i.e.*, sharply defined absorption-lines) corresponds to an important difference in the intrinsic luminosities of the stars. Hitherto, however, it has

been usual to accept the Draper classification as at least the most complete available for our investigations.

Colour-Index.—Stars may be classified according to colour as an alternative to spectral type. Both methods involve dividing the stars according to the nature of the light emitted by them, and thus have something in common. Perhaps we might not expect a very close correspondence between the two classifications; for, whilst colour depends mainly on the continuous background of the spectrum, the spectral type is determined by the fine lines and bands, which can have little direct effect on the colour. Nevertheless a close correlation is found between the two characters, owing no doubt to the fact that both are intimately connected with the effective temperature of the star.

The most convenient measure of colour is afforded by the difference, photographic *minus* visual magnitude; this is called the *colour-index.* The relation between the spectral type and the colour-index is shown below.

Spectral Type.	Colour index according to	
	King.	Schwarzschild.
	m	m
Bo	−0·31	−0·64
Ao	0·00	−0·32
Fo	+0·32	0·00
Go	+0·71	+0·32
Ko	+1·17	+0·95
M	+1·68	+1·89

King's results [3] refer to the Harvard visual and photographic scales; Schwarzschild's [4] to the Göttingen photographic and Potsdam visual determinations. Allowing for the constant difference, depending on the particular type of spectrum for which the photographic and visual magnitudes are

made to agree, the two investigations confirm one another closely.

The foregoing results are derived from the means of a considerable number of stars, but the Table may be applied to individual stars with considerable accuracy. Thus the spectral type can be found when the colour-index is known, and conversely. At least in the case of the early type stars, the spectral type fixes the colour-index with an average uncertainty of not more than $0^{m}\cdot1$; for Types G and K larger deviations are found, but the correlation is still a very close one.

Yet another method of classifying stars according to the character of the light emitted is afforded by measures of the "effective wave-length." If a coarse grating, consisting of parallel strips or wires equally spaced, is placed in front of the object-glass of a telescope, diffraction images appear on either side of the principal image. These diffraction images are strictly spectra, and the spot which a measurer would select as the centre of the image will depend on the distribution of intensity in the spectrum. Each star will thus have a certain effective wave-length which will be an index of its colour, or rather of the appreciation of its colour by the photographic plate. For the same telescope and grating the interval between the two first diffracted images is a constant multiple of the effective wave-length. The method was first used by K. Schwarzschild in 1895; and an important application of it was made by Prosper Henry to determine the effect of atmospheric dispersion on the places of the planet Eros. It has been applied to stellar classification by E. Hertzsprung.[5]

Parallax.—The annual motion of the earth around the Sun causes a minute change in the direction in which a star is seen, so that the star appears to describe a small ellipse in the sky. This periodic displacement is superposed on the uniform proper-motion of the star,

which is generally much greater in amount; there is, however, no difficulty in disentangling the two kinds of displacement. Since we are only concerned with the direction of the line joining the two bodies, the effect of the earth's motion is the same as if the earth remained at rest, and the star described an orbit in space equal to that of the earth, but with the displacement reversed, so that the star in its imaginary orbit is six months ahead of the earth. This orbit is nearly circular; but, as it is generally viewed at an angle, it appears as an ellipse in the sky. In any case the major axis of the ellipse is equal to the diameter of the earth's orbit; and, since the latter length is known, a determination of its apparent or angular magnitude affords a means of calculating the star's distance. The parallax is defined as the angle subtended by one astronomical unit (the radius of the earth's orbit) at the distance of the star, and is equivalent to the major semiaxis of the ellipse which the star appears to describe.

The measurement of this small ellipse is always made relatively to some surrounding stars, for it is hopeless to make absolute determinations of direction with the necessary accuracy. The relative parallax which is thus obtained needs to be corrected by the amount of the average parallax of the comparison stars in order to obtain the absolute parallax. This correction can only be guessed from our general knowledge of the distances of stars similar in magnitude and proper motion to the comparison stars; but as it could seldom be more than $0''\cdot01$, not much uncertainty is introduced into the final result from this cause.

The parallax is the most difficult to determine of all the quantities which we wish to know, and for only a very few of the stars has it been measured with any approach to certainty. Until some great advance is made in means of measurement, all but a few hundreds of the nearest stars must be out of range of the method. But so

laborious are the observations required, that even these will occupy investigators for a long while. In general, the published lists of parallaxes contain many that are extremely uncertain, and some that are altogether spurious. Statistical investigations based on these are liable to be very misleading. Nevertheless, it is believed that by rejecting unsparingly all determinations but those of the highest refinement, some important information can be obtained, and in Chapter III. these results are discussed. In addition, determinations which are not individually of high accuracy may be used for finding the mean parallaxes of stars of different orders of magnitude and proper motion, provided they are sensibly free from systematic error; these at least serve to check the results found by less direct methods.

The measurements are generally made either by photography or visually with a heliòmeter. The former method now appears to give the best results owing to the greater focal-length of the instruments available. It has also the advantage of using a greater number of comparison stars, so that there is less chance of the correction to reduce to absolute parallax being inaccurate. Some early determinations with the heliometer are, however, still unsurpassed. The meridian-circle is also used for this work, and considerable improvement is shown in the more recent results of this method; but we still think that meridian parallaxes are to be regarded with suspicion, and have deemed it best not to use them at all in Chapter III.

A convenient unit for measuring stellar distances is the *parsec*,* or distance which corresponds to a parallax of one

* The *parsec*, a portmanteau-name suggested by H. H. Turner, will be used throughout this book. Several different units of stellar distance have, however, been employed by investigators. Kobold's *Sternweite* is identical with the parsec. Seeliger's *Siriusweite* corresponds to a parallax 0″·2, and Charlier's *Siriometer* to a million astronomical units or parallax 0″·206. The light-year which, notwithstanding its inconvenience and irrelevance, has sometimes crept from popular use into technical investigations, is equal to 0·31 parsecs.

second of arc. This is equal to 206,000 astronomical units or about 19,000,000,000,000 miles. The nearest fixed star, α Centauri, is at a distance of 1·3 parsecs. There are perhaps thirty or forty stars within a distance of five parsecs, and, of course, the number at greater distances will increase as the cube of the limiting distance, so long as the distribution is uniform. But these nearest stars are not by any means the brightest visible to us; the range in intrinsic luminosity is so great that the apparent magnitude is very little clue to the distance. A sphere of radius thirty parsecs would probably contain 6000 stars; but the 6000 stars visible to the average eye are spread through a far larger volume of space. It appears indeed that some of the naked-eye stars are situated in the remotest parts of the stellar system.

A parallax-determination may be considered first-class if its probable error is as low as $0''\cdot01$. If, for instance, the measured parallax is $0''\cdot05 \pm 0''\cdot01$, it is an even chance that the true value lies between $0''\cdot06$ and $0''\cdot04$, and we probably should not place much confidence in any nearer limits than $0''\cdot07$ and $0''\cdot03$. This is equivalent to saying that the star is distant something between fourteen and thirty-three parsecs from us. It will be seen that, when the parallax is as low as $0''\cdot05$, even the best measures give only the very roughest idea of the distance of the star, and for smaller values the information becomes still more vague. Clearly, to be of value a parallax must be at least $0''\cdot05$. It may be estimated that there are not more than 2000 stars so near as this, and a very large proportion of these will be fainter than the tenth magnitude. The chances are that, of five plates of the international Carte du Ciel taken at random, only one will be fortunate enough to pick up a serviceable parallax, and even that is likely to be a very inconspicuous star, which would evade any but the most thorough search. The prospect of so overwhelming a proportion of negative results suggests that, for the present

at any rate, work can be most usefully done on special objects for which an exceptionally large proper motion affords an *a priori* expectation of a sensible parallax. A star of parallax 0″·05 may be expected to have a proper motion of 20″ per century or more, and that seems to be a reasonable limit to work down to.

It will generally be admitted that a most valuable extension of our knowledge will result from precise measures of the distances of as many as possible of the individual stars that come within the range above mentioned. But many investigators have also sought to determine the mean parallaxes of stars of different magnitudes or motions. When the individual distances are too uncertain, the means of a large number may still have some significance. Whilst some useful results can be and have been obtained by this kind of research, its possibilities seem to be very limited. Generally speaking, this class of determination requires even greater refinement than the measurement of individual parallaxes; refinement which is scarcely yet within reach. For example, the mean parallax of stars of the sixth magnitude is 0″·014 (perhaps a rather high estimate); that of the comparison stars would probably be about half this, so that the relative parallax actually measured would be 0″·007. The possible systematic errors depending on magnitude and colour (the mean colour of the sixth magnitude is perhaps different from the ninth) make the problem of determining this difference one of far higher difficulty than that of measuring the parallax of an individual star. It means gaining almost another decimal place beyond the point yet reached. We need not dwell on the enormous labour of observing the necessary fifty or one hundred sixth magnitude stars to obtain this mean with reasonable accuracy; it might well be thought worth the trouble; but there is no evidence that systematic errors have as yet been brought as low as 0″·001

even in the best work, and indeed it seems almost inconceivable.

From these considerations it appears that parallax-determinations should be directed towards:

(1) Individual stars with proper motions exceeding 20″ a century. This will yield many negative results, but a fair proportion of successes.

(2) Classes of stars with proper motions less than 20″, but still much above the average. These parallaxes will have to be found individually, but for the most part only the mean result for a class will be of use.

(3) A possible extension to classes of stars not distinguished by large proper motion, provided it is realised that a far higher standard is required for this work, and that a freedom from systematic error as great as 0″·001 can be ensured.

Proper Motion.—For stellar investigation the proper motions, *i.e.*, the apparent angular motions of the stars, form most valuable material. For extension in our knowledge of magnitudes, parallaxes, radial velocities, and spectral classification, we shall ultimately come to depend on improved equipment and methods of observation; but the mere lapse of time enables the proper motions to become known with greater and greater accuracy, and the only limit to our knowledge is the labour that can be devoted, and the number of centuries we are content to wait.

Proper motions of stars differ greatly in amount, but in general the motion of any reasonably bright star (*e.g.*, brighter than $7^{m}\cdot0$) is large enough to be detected in the time over which observations have already accumulated. Whilst it is quite the exception for a star to have a measurable parallax, it is exceptional for the proper motion to be insensible. It may be useful to give some idea of the certainty and trustworthiness of the proper motions in ordinary use, though the figures are necessarily only

approximate. When the probable error is about 1″ per century in both coordinates, the motion may be considered to be determined fairly satisfactorily; the Groombridge and Carrington Catalogues, largely used in statistical investigations, are of about this order of accuracy. A higher standard—probable error about 0″·6 per century—is reached in Boss's "Preliminary General Catalogue of 6188 Stars," which is much the best source of proper motions at present available. For some of the fundamental stars regularly observed at a large number of places throughout the last century the accidental error is as low as 0″·2 per century; but the inevitable systematic errors may well make the true error somewhat larger. Various sources of systematic error, particularly uncertainties in the constant of precession and the motion of the equinox, may render the motions in any region of the sky as much as 0″·5 per century in error; it is unlikely that the systematic error of the best proper motions can be greater than this, except in one or two special regions of southern declination, where exceptional uncertainty exists.

We may thus regard the proper motions used in statistical researches as known with a probable error of not more than 1″ per century in right ascension and declination. Roughly speaking, an average motion is from 3″ to 7″ per century. A centennial motion of more than 20″ is considered large, although there are some stars which greatly exceed this speed. The fastest of all is C.Z. 5^h 243—a star of the ninth magnitude found by J. C. Kapteyn and R. T. A. Innes on the plates of the Cape Photographic Durchmusterung—which moves at the rate of 870″ per century. This speed would carry it over an arc equal to the length of Orion's belt in just above a thousand years. Table 1 shows the stars at present known of which the centennial speed exceeds 300″. The number of faint stars on this list is very striking; and, as our information practically stops at the ninth magnitude, it

may be conjectured that there are a number of still fainter stars yet to be detected.

TABLE 1.

Stars with large Proper Motion.

Name.	R.A. 1900.	Dec. 1900.	Annual Proper Motion.	Magnitude.
	h. m.	°	"	
C.Z. 5^h 243	5 8	−45·0	8·70	8·3
Groombridge 1830	11 47	+38·4	7·07	6·5
Lacaille 9352	22 59	−36·4	7·02	7·4
Cordoba 32416	0 0	−37·8	6·07	8·5
61^1 Cygni	21 2	+38·3	5·25	5·6
Lalonde 21185	10 58	+36·6	4·77	7·6
ϵ Indi	21 56	−57·2	4·67	4·7
Lalonde 21258	11 0	+44·0	4·46	8·9
o^2 Eridani	4 11	− 7·8	4·08	4·5
O.A. (s.) 14318	15 5	−16·0	3·76	9·6
O.A. (s.) 14320	15 5	−15·9	3·76	9·2
μ Cassiopeiae	1 2	+54·4	3·75	5·3
a^1 Centauri	14 33	−60·4	3·66	0·3
Lacaille 8760	21 11	−39·2	3·53	7·3
e Eridani	3 16	−43·4	3·15	4·3
O.A. (N.) 11677	11 15	+66·4	3·03	9·2

Comparatively little is known of the motions of stars fainter than the ninth magnitude. The Carrington proper motions, discussed by F. W. Dyson, carry us down to $10^m\cdot3$ for the region within 9° of the North Pole. A number of the larger proper motions of faint stars in the Oxford Zone of the Carte du Ciel have been published by H. H. Turner and F. A. Bellamy.[6] Further, by the reduction of micrometric measurements, G. C. Comstock[7] has obtained a number of proper motions extending even to the thirteenth magnitude. There is no difficulty to be expected in securing data for faint stars; but the work has been taken up comparatively recently, and the one essential is—lapse of a sufficient time.

Radial Velocity.—The velocity in the line of sight is measured by means of a spectroscope. In accordance with Doppler's Principle, the lines in the spectrum of a star are displaced towards the red or the violet

(relatively to a terrestrial comparison spectrum) according as the star is receding from or approaching the earth. Unlike the proper motion, the radial motion is found directly in kilometres per second, so that the actual linear speed, unmixed with the doubtful element of distance, is known. Hitherto it has scarcely been possible to measure the velocities of stars fainter than the fifth magnitude, but that limitation is now being removed. The main difficulty in regard to the use of the results is the large proportion of spectroscopic binary stars, about one in three or four of the total number observed. As the orbital motion is often very much larger than the true radial velocity, it is essential to allow sufficient time to elapse to detect any variation in the motion, before assuming that the measures give the real secular motion, of which we are in search. Another uncertainty arises from possible systematic errors affecting all the stars belonging to a particular type of spectrum. There is reason to believe that the measured velocity of recession of the Type B stars is systematically 5 km. per sec. too great.[8] This may be due to errors in the standard wave-lengths employed, or to a pressure-shift of the lines under the physical conditions prevailing in this kind of star. Smaller errors affect the stars of other types.

Apart from possible systematic error, a remarkable accuracy has been attained in these observations. For a star with sharp spectral lines a probable error of under 0·25 km. per sec. is well within reach. Stars of Types B and A have more diffuse lines and the results are not quite so good; but the accuracy even in these cases is far beyond the requirements of the statistician. The observed velocities range up to above one hundred km. per sec., but speeds greater than sixty km. per sec. are not very common. The greatest speed yet measured is that of Lalande 1966, viz., 325 km. per sec. The next highest is C.Z.5^h 243, already mentioned as having the greatest apparent motion across

the sky; it is observed to be receding at the rate of 242 km. per sec., or 225 km. per sec. if we make allowance for the Sun's own motion. As these figures refer to only one component of motion, the total speeds of stars are sometimes considerably greater.

Radial motions of about 1400 stars have now been published, the great bulk of the observations having been made at the Lick Observatory. Most of this material only became accessible to investigators in 1913, and there has scarcely yet been time to make full use of the new data.

There are a few systems which can be observed both as visual and as spectroscopic binaries. In such cases it is possible to deduce the distance of the star by a method quite independent of the usual parallax determinations. From the visual observations, the period and the other elements of the orbit can be found. The dimensions, however, are all expressed in arc, *i.e.*, in linear measure divided by the unknown distance of the star. From these elements we can calculate for any date the relative velocity in the line of sight of the two components; but this also will be expressed as a linear velocity divided by the unknown distance. By comparing this result with the same relative velocity measured spectrographically, and therefore directly in linear measure, the distance of the system can be derived. This method is of very limited application; but in the case of α Centauri it has given a very valuable confirmation of the parallax determined in the ordinary way. It increases our confidence that the usual method of measuring stellar distances is a sound one.

Mass and Density.—Knowledge of the masses and densities of stars is derived entirely from binary systems. The sources of information are of three kinds:—

(1) From visual binaries.

(2) From ordinary spectroscopic binaries.

(3) From eclipsing variables.

The combined mass of the two components of a binary system can be found from the length of the major semi-axis of the orbit a and the period P by the formula

$$m_1 + m_2 = a^3/P^2.$$

Here the masses are expressed in terms of the Sun's mass as unit, and the astronomical unit and the year are taken as the units of length and time.

In a well-observed visual orbit, all the elements are known (except for an ambiguity of sign of the inclination), but the major axis is expressed in arc. This can be converted into linear measure, if the parallax has been determined; and hence $m_1 + m_2$ can be found. When further, besides the relative orbit, a rough absolute orbit of one of the components has been found, by meridian observations or otherwise, the ratio m_1/m_2 is determinable, and m_1 and m_2 are deduced separately. Owing to the difficulty of determining parallaxes, cases of a complete solution of this kind are rare. They are, however, sufficient to indicate the fact that the range in the masses of the stars is not at all proportionate to the huge range in their luminosities.

TABLE 2.

Well-determined Masses of Stars.

Star	Combined System				Brighter Component	
	Mass (Sun = 1)	Period Years	a	Parallax	Luminosity (Sun = 1)	Spectral Type
			"	"		
ζ Herculis . .	1·8	34·5	1·35	0·14	5·0	G
Procyon . . .	1·3	39·0	4·05	0·32	9·7	F5
Sirius	3·4	48·8	7·59	0·38	48·0	A
α Centauri . .	1·9	81·2	17·71	0·76	2·0	G
70 Ophiuchi .	2·5	88·4	4·55	0·17	1·1	K
o^2 Eridani .	0·7	180·0	4·79	0·17	0·84	G
η Cassiopeiae* .	1·0	328·0	9·48	0·20	1·4	F5 8

* Another published orbit $P = 508^y$ $a = 12{\cdot}2''$ gives the mass $= 0{\cdot}9$. The great uncertainty of the orbit appears to have little effect on the result.

Table 2 contains all the systems, of which the masses can be ascertained with reasonable accuracy, *i.e.*, systems for which good orbits [9] and good parallax-determinations [10] have been published. Possibly some of the more doubtful orbits would have been good enough for the purpose, but I doubt if the list could be much extended without lowering the standard.

Another fact which appears is that the ratio of the masses of the two components of a binary is generally not far from equality, notwithstanding considerable differences in the luminosity. Thus Lewis Boss [11] in ten systems found that the ratio of the mass of the faint star to the brighter star ranged from 0·33 to 1·1,* the mean being 0·71. The result is confirmed by observations of such spectroscopic binaries as show the lines of both components, though in this case the disparity of luminosity cannot be so great.

Even when the parallax is not known, important information as to the density can be obtained. Consider for simplicity a system in which one component is of negligible mass; the application to the more general case requires only slight modifications, provided m_1/m_2 is known or can be assumed to have its average value.

Let

d be the distance of the star
b its radius
S its surface brightness
L, l its intrinsic and apparent luminosities
M its mass
ρ its density
γ the constant of gravitation

Then

$$M=\frac{4\pi^2}{\gamma}\frac{a^3}{P^2}$$

$$l=\frac{L}{d^2}$$

$$L=\pi b^2 S$$

and

$$M=\tfrac{4}{3}\pi\rho b^3$$

* Excluding one very doubtful result.

From these

$$\rho = S^{\frac{3}{2}} \times \frac{3\pi}{\gamma}\left(\frac{a}{d}\right)^3 \cdot \frac{1}{P^2}\left(\frac{\pi}{l}\right)^{\frac{3}{2}}$$

$\frac{a}{d}$ is the semi-axis of the orbit in arc, and l and P are observed quantities; consequently the coefficient of $S^{\frac{3}{2}}$ is known. We have thus an expression for the density in terms of the surface brightness, and can at least compare the densities of those stars which, on spectroscopic evidence, may be presumed to have similar surface conditions.

The density is found to have a large range, many of the stars being apparently in a very diffused state with densities perhaps not greater than that of atmospheric air.

The spectroscopic binaries also give some information as to the masses of stars. The formula $(m_1 + m_2) = a^3/P^2$ is applicable, and as a is now found in linear measure it is not necessary to know the parallax. The quantity deduced from the observations is, however, in this case $a \sin i$, where i is the inclination of the plane of the orbit. The inclination remains unknown, except when the star is an eclipsing variable * or in the rare case when the system is at the same time a visual and a spectroscopic binary. For statistical purposes, such as comparing the masses of different types of stars, we may assume that in the mean of a sufficient number of cases the planes of the orbits will be distributed at random, and can adopt a mean value for $\sin i$. Thus from spectroscopic binaries the average masses of classes, but not of individual stars, can be found.

In the case of eclipsing variable stars, the densities of the two components can be deduced entirely from the light-curve of the star. Although these are necessarily spectroscopic binary systems, observations of the radial velocity are not needed, and are not used in the results. The actual procedure, which is due to H. N. Russell and H. Shapley,[12] is too complicated to be detailed here, as

* In this case it is evident that i must be nearly 90°, and accordingly $\sin i$ may be taken as unity.

it bears only incidentally on our subject; but the general principle may be briefly indicated. It will be easily realised that the proportionate duration of the eclipse and other features of the light curve do not depend on the absolute dimensions of the system, but on the ratio of the three linear quantities involved, viz., the diameters of the two stars and the distance between their centres. By strictly geometrical considerations therefore, we find the radii r_1, r_2 of the stars expressed in terms of the unknown semi-axis of the orbit a as unit. Now the relation between the mass and density of a star involves the cube of the radius, and the dynamical relation between the mass and the period involves the cube of a. Thus, on division, the absolute masses and the unknown unit a disappear simultaneously, and we are left with the density expressed in terms of the period and the known ratios $\frac{r_1}{a}$, $\frac{r_2}{a}$. The key to the solution is that in astronomical units the *Dimensions* of density are $(\text{time})^{-2}$; the density thus depends on the period, and on the ratios, but not the absolute values, of the other constants of the system.

The densities found in this way are not quite rigorously determined. It is necessary to assume a value of the ratio of the masses of the two stars; as already explained, this ratio does not differ widely from unity, but in extreme cases the results may be as much as fifty per cent. in error from this cause. Further, the darkening at the limb of the star has some effect on the determination, and the assumed law of darkening is hypothetical. By taking different assumptions, between which the truth is bound to lie, it can be shown that these uncertainties do not amount to anything important, when regard is had to the great range in stellar densities which is actually found.

We may conclude this account of the nature of the observations, on which our knowledge of the stellar

universe is based, by a reference to J. C. Kapteyn's "Plan of Selected Areas." When the study of stars was confined mainly to those brighter than the seventh magnitude, and again when it was extended as far as the ninth, or tenth, a complete survey of all the stars was not an impossible aim, and indeed all the data obtainable could well be utilised. But investigations are now being pushed towards the fifteenth and even lower magnitudes. These fainter stars are so numerous that it is impossible and unnecessary to do more than make a selection. As the different kinds of observation for parallax, proper motion, magnitude, spectral type, and radial velocity are highly specialised and usually carried out at different observatories, some co-operation is necessary in order that so far as possible the observations may be concentrated on the same groups of stars. Kapteyn's [13] plan of devoting attention to 206 selected areas, distributed all over the sky, so as to cover all varieties of stellar distribution, has met with very general support. The areas have their centres on or near the circles of declination, 0°, ± 15°, ± 30°, ± 45°, ± 60°, ± 75°, ± 90°. The exact centres have been chosen with regard to various practical considerations; but the distribution is very nearly uniform. In addition to the main "Plan," 46 areas in the Milky Way have been chosen, typical of its main varieties of structure. The area proper consists of a square 75′ × 75′, or alternatively a circle of 42′ radius; but the dimensions may be extended or diminished for investigations of particular data.

The whole scheme of work includes nine main subdivisions: (1) A Durchmusterung of the areas. (2) Standard photographic magnitudes. (3) Visual and photovisual magnitudes. (4) Parallaxes. (5) Differential proper motions. (6) Standard proper motions. (7) Spectra. (8) Radial velocities. (9) Intensity of the background of the sky. The Durchmusterung is well advanced; it will include all stars to 17^m, the positions being given with a

probable error of about 1″ in each co-ordinate, and the magnitudes (differential so far as this part of the work is concerned) with a probable error of $0^{m}\cdot1$. Considerable progress has been made with the determination of sequences of standard photographic magnitudes for each area. The work of determining the visual magnitudes has been partly accomplished for the northern zones. For parallaxes, most of the areas have been portioned out between different Observatories; the greatest progress has been made at the Cape Observatory for the southern sky, but the plates have not yet been measured. From what has already been stated with regard to the practical possibilities of parallax-determinations, it will be seen that there is some doubt as to the utility of this part of the Plan. For the proper motions of faint stars, the work has necessarily been confined mainly to obtaining plates for the initial epoch. At the Radcliffe Observatory, 150 plates have been stored away undeveloped, ready for a second exposure after a suitable interval, but in most cases it is intended to rely on the parallax plates for giving the initial positions. For standard proper motions in the northern sky, observations are shortly to be started at Bonn; these will serve for comparison with older catalogues, but they may also be regarded as initial observations for more accurate determinations in the future. Determinations of spectral type as far as the ninth magnitude, made at Harvard, will shortly be available for these areas and, indeed, for the whole sky. The extension to the eleventh magnitude is very desirable, and is one of the most urgent problems of the whole Plan. Radial velocity determinations are being pressed as far as $8^{m}\cdot0$ at Mount Wilson, but rapid progress is not to be expected until the completion of the 100-inch reflector. A valuable, though unofficial, addition to the programme is E. A. Fath's Durchmusterung[14] of all the nebulæ in the areas from the North pole to Dec. −15°.

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11. Boss, *Preliminary General Catalogue of 6188 Stars*, Introduction, p. 23.
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A useful discussion of the methods of determining photographic magnitude will be found in an R.A.S. "Council Note," *Monthly Notices*, Vol. 73, p. 291 (1913).

Spectral Types.—The most extensive determinations are to be found in *Harv. Ann.*, Vol. 50, which gives the type for stars brighter than $6^m{\cdot}5$. Scattered determinations of many fainter stars also exist. It is understood that a very comprehensive catalogue containing the types of 200,000 stars will shortly be issued from Harvard.

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Slocum and Mitchell, *Astrophysical Journal*, Vol. 38, p. 1.

Very useful compilations of the parallaxes taken from all available sources are given by Kapteyn and Weersma, *Groningen Publications*, No. 24, and by Bigourdan, *Bulletin Astronomique*, Vol. 26. Fuller references are given in these publications.

Proper Motions.—Lewis Boss's *Preliminary General Catalogue of* 6188 *Stars*, which includes proper motions of all the brighter stars, supersedes many earlier collections.

Dyson and Thackeray's *New Reduction of Groombridge's Catalogue* contains 4243 stars, including many fainter than the eighth magnitude, within 50° of the North Pole.

Still fainter stars are contained in the Greenwich–Carrington proper motions discussed by Dyson. Some of these are published in the *Second Nine-Year Catalogue* (1900), but certain additional corrections are required to the motions there given (see *Monthly Notices*, Vol. 73, p. 336). The complete list (unpublished) contains 3735 stars.

Porter's Catalogue, *Cincinnati Publications*, No. 12, gives 1340 stars of especially large proper motion.

Radial Velocities.—Catalogues containing a practically complete summary of the radial velocities at present available for discussion are given by Campbell in *Lick Bulletin*, Nos. 195, 211, and 229. These contain about 1350 stars.

CHAPTER II

GENERAL OUTLINE

THIS chapter will be devoted to a general description of the sidereal universe as it is revealed by modern researches. The evidence for the statements now made will appear gradually in the subsequent part of the book, and minor details will be filled in. But it seems necessary to presume a general acquaintance with the whole field of knowledge before starting on any one line of detailed investigation. At first sight it might seem possible to divide the subject into compartments—the distribution of the stars through space, their luminosities, their motions, and the characters of the different spectral types—but it is not possible to pursue these different branches of inquiry independently. Any one mode of investigation leads, as a rule, to results in which all these matters are involved together, and no one inquiry can be worked out to a conclusion without frequent reference to parallel investigations. We have therefore adopted the unusual course of placing what may be regarded as a summary thus early in the book.

In presenting a summary, we may claim the privilege of neglecting many awkward difficulties and uncertainties that arise, promising to deal fairly with them later. We can pass over alternative explanations, which for the moment are out of favour; though they need to be kept

alive, for at any moment new facts may be found, which will cause us to turn to them again. The bare outline, devoid of the details, must not be taken as an adequate presentation of our knowledge, and in particular it will fail to convey the real complexity of the phenomena discussed. Above all, let it be remembered that our object in building up a connected idea of the universe from the facts of observation is not to assert as unalterable truth the views we arrive at, but, by means of working hypotheses, to assist the mind to grasp the interrelations of the facts, and to prepare the way for a further advance. When we look back on the many transformations that theories in all departments of science have undergone in the past, we

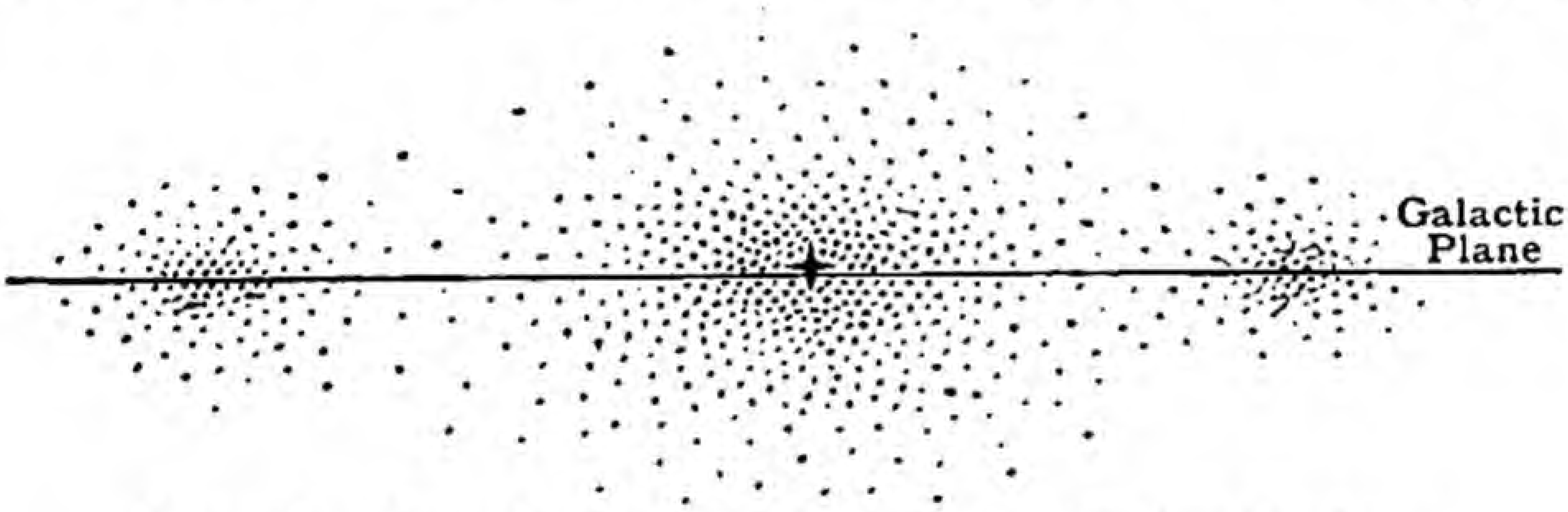

FIG. 1.—Hypothetical Section of the Stellar System.

shall not be so rash as to suppose that the mystery of the sidereal universe has yielded almost at the first attack. But as each revolution of thought has contained some kernel of surviving truth, so we may hope that our present representation of the universe contains something that will last, notwithstanding its faulty expression.

It is believed that the great mass of the stars with which we are concerned in these researches are arranged in the form of a lens- or bun-shaped system. That is to say, the system is considerably flattened towards one plane. A general idea of the arrangement is given in Fig. 1, where the middle patch represents the system to which we are now referring. In this aggregation the Sun occupies a fairly central position, indicated by +. The median plane of the lens is the same as the plane marked out in the sky by the

Milky Way, so that, when we look in any direction along the galactic plane (as the plane of the Milky Way is called), we are looking towards the perimeter of the lens where the boundary is most remote. At right angles to this, that is, towards the north and south galactic poles, the boundary is nearest to us; so near, indeed, that our telescopes can penetrate to its limits. The actual position of the Sun is a little north of the median plane; there is little evidence as to its position with respect to the perimeter of the lens; all that we can say is that it is not markedly eccentric.

The thickness of the system, though enormous compared with ordinary units, is not immeasurably great. No definite distance can be specified, because it is unlikely that there is a sharp boundary; there is only a gradual thinning out of the stars. The facts would perhaps be best expressed by saying that the surfaces of equal density resemble oblate spheroids. To give a general idea of the scale of the system, it may be stated that in directions towards the galactic poles the density continues practically uniform up to a distance of about 100 parsecs; after that the falling off becomes noticeable, so that at 300 parsecs it is only a fraction (perhaps a fifth) of the density near the Sun. The extension in the galactic plane is at least three times greater. These figures are subject to large uncertainties.

It seems that near the Sun the stars are scattered in a fairly uniform manner; any irregularities are on a small scale, and may be overlooked in considering the general architecture of the stellar system. But in the remoter parts of the lens, or more probably right beyond it, there lies the great cluster or series of star-clouds which make up the Milky Way. In Fig. 1 this is indicated (in section) by the star-groups to the extreme left and right. These star-clouds form a belt stretching completely round the main flattened system, a series of irregular agglomera-

tions of stars of wonderful richness, diverse in form and grouping, but keeping close to the fundamental plane. It is important to distinguish clearly the two properties of the galactic plane, for they have sometimes been confused. First, it is the median plane of the bun-shaped arrangement of the nearer stars, and, secondly, it is the plane in which the star-clouds of the Milky Way are coiled.

Not all the stars are equally condensed to the galactic plane. Generally speaking, the stars of early type congregate there strongly, whereas those of late types are distributed in a much less flattened, or even in a practically globular, form. The mean result is a decidedly oblate system; but if, for example, we consider separately the stars of Type M, many of which are at a great distance from us, they appear to form a nearly spherical system.

In the Milky Way are found some vast tracts of absorbing matter, which cut off the light of the stars behind. These are of the same nature as the extended irregular nebulæ, which are also generally associated with the Milky Way. The dark absorbing patches and the faintly shining nebulæ fade into one another insensibly, so that we may have a dark region with a faintly luminous edge. Whether the material is faintly luminous or not, it exercises the same effect in dimming or hiding the bodies behind it. There is probably some of this absorbing stuff even within the limits of the central aggregation. In addition to these specially opaque regions, it is probable that fine particles may be diffused generally through interstellar space, which would have the effect of dimming the light of the more distant stars; but, so far as can be ascertained, this "fog" is not sufficient to produce any important effect, and we shall usually neglect it in the investigations which follow.

In studying the movements of the stars we necessarily leave the remoter parts of space, confining attention mainly to the lens-shaped system, and perhaps only to the inner

parts of it, where the apparent angular movements are appreciable. Researches on radial motions need not be quite so limited, because in them the quantity to be measured is independent of the distance of the stars; but here too the nearer parts of the system obtain a preference, for observations are confined to the bright stars. Although thus restricted, our sphere of knowledge is yet wide enough to embrace some hundreds of thousands of stars (considered through representative samples); the results that are deduced will have a more than local importance.

The remarkable result appears that within the inner system the stars move with a strong preference in two opposite directions in the galactic plane. There are two favoured directions of motion; and the appearance is as though two large aggregates of stars of more or less independent origin were passing through one another, and so for the time being were intermingled. It is true that such a straightforward interpretation seems to be at variance with the plan of a single oblate system, which has just been sketched. Various alternatives will be considered later; meanwhile it is sufficient to note that the difficulty exists. But, whatever may be the physical cause, there is no doubt that one line in the galactic plane is singled out, and the stars tend to move to and fro along it in preference to any transverse directions. We shall find it convenient to distinguish the two streams of stars, which move in opposite directions along the line, reserving judgment as to whether they are really two independent systems or whether there is some other origin for this curious phenomenon. The names assigned are

Stream I. moving towards R.A. 94°, Dec. +12°.
,, II. ,, ,, R.A. 274°, Dec. −12°.

The relative motion of one stream with respect to the other is about 40 km. per sec.

The Sun itself has an individual motion with respect to the mean of all the stars. Its velocity is 20 kilometres per second directed towards the point R.A. 270° Dec. + 35°. The stellar movements that are directly observed are referred to the Sun as standard, and are consequently affected by its motion. This makes a considerable alteration in the apparent directions of the two streams; thus we find

Stream I. moving towards R.A. 91°, Dec. $-15°$ } relatively to
,, II. ,, ,, R.A. 288°, Dec. $-64°$ } the Sun;

and moreover the velocity of the first stream is about 1·8 times that of the second (probably 34 and 19 km. per sec., respectively). Stream I is sometimes therefore referred to as the quick-moving stream, and Stream II as the slow-moving one; but it must be remembered that this description refers only to the motion relative to the Sun. The stars which constitute the streams have, besides the stream-motion, individual motions of their own; but the stream-motion sufficiently dominates over these random motions to cause a marked general agreement of direction.

Stream I contains more stars than Stream II in the ratio 3 : 2. Though this ratio varies irregularly in different parts of the sky, the mixture is everywhere fairly complete. Moreover, there is no appreciable difference in the average distances of the stars of the two streams. It is not a case of a group of nearer stars moving in one direction across a background of stars moving the opposite way; there is evidence that the two streams thoroughly permeate each other at all distances and in all parts of the heavens.

A more minute investigation of this phenomenon shows that it is complicated by differences in the behaviour of stars according to their spectral type. An analysis which treats the heterogeneous mass of the stars as a whole

without any separation of the different types will fail to give a complete insight into the phenomenon. But, until a great deal more material is accumulated, this interrelation of stream motions and spectral type cannot be worked out very satisfactorily. The outstanding feature, however, is that the stars of the Orion Type (Type B) seem not to share to any appreciable extent in the star-streaming tendency. Their individual motions, which are always very small, are nearly haphazard, though the apparent motions are, of course, affected by the solar motion. They thus form a third system, having the motion of neither of the two great streams, but nearly at rest relatively to the mean of the stars. This third system is not entirely confined to the B stars. In the ordinary analysis into two streams we always find some stars left over—comparatively few in number yet constituting a distinct irregularity—which evidently belong to the same system. These stars may be of any of the spectral types. There is something arbitrary in this dissection into streams (which may be compared to a Fourier or spherical harmonic analysis of observations), and we can, if we like, adopt a dissection which gives much fuller recognition to this third system.

At one time it seemed that the third stream, Stream O as it is called, might be constituted of the very distant stars, lying beyond those whose motions are the main theme of discussion. If that were so, it would not be surprising to find that they followed a different law, and were not comprised in the two main streams. But this explanation is now found to be at variance with the facts. We have to recognise that Stream O is to be found even among the nearer stars.

The smallness of the individual movements of the B stars is found to be part of a much more general law. Astrophysicists have by a study of the spectra arranged the stars in what they believe to be the successive stages

of evolution. Now it is found that there is a regular progression in the size of the linear motions from the youngest to the oldest stars. It is as though a star was born without motion, and gradually acquires or grows one. The average individual motion (resolved in one direction) increases steadily from about 6·5 km. per sec. for Type B to 17 km. per sec. for Type M.

We may well regard this relation of age and velocity as one of the most startling results of modern astronomy. For the last forty years astrophysicists have been studying the spectra and arranging the stars in order of evolution. However plausible may be their arguments one would have said that their hypotheses must be for ever outside the possibility of confirmation. Yet, if this result is right, we have a totally distinct criterion by which the stars are arranged in the same order. If it is really true that the mean motion of a class of stars measures its progress along the path of evolution, we have a new and powerful aid to the understanding of the steps of stellar development.

It is not at all easy to explain why the stellar velocities increase with advancing development. I am inclined to think that the following hypothesis offers the best explanation of the facts. In a primitive state the star-forming material was scattered much as the stars are now, that is, densely along the galactic plane up to moderate distances, and more thinly away from the plane and at great distances. Where the material was rich, large stars, which evolved slowly, were formed; where it was rare, small stars, which developed rapidly. The former are our early type stars; not having fallen in from any great distance they move slowly and in the main parallel to the galactic plane. The latter—our late type stars—have been formed at a great distance, and have acquired large velocities in falling in; moreover since they were not necessarily formed near the galactic plane, their motions are not so predominantly parallel to it.

Whilst the individual motion of a star gradually increases from type to type, the stream-motion appears with remarkable suddenness. Throughout Type B, even up to B 8 and B 9, the two star-streams are unperceived; but in the next type, A, the phenomenon is seen in its clearest and most pronounced form. Through the remaining types it is still very prominent, but there is an appreciable falling off. There is no reason to believe that this decline is due to any actual decrease in the stream-velocities; it is only that the gradual increase of the haphazard motions renders the systematic motions less dominant.

Among the most beautiful objects that the telescope reveals are the star-clusters, particularly the globular clusters, in which hundreds or even thousands of stars are crowded into a compact mass easily comprised within the field of a telescope. Recent research has revealed several systems, presumably of a similar nature to these, which are actually in our neighbourhood, and in one case even surrounding us. Being seen from a short distance the concentration is lost, and the cluster scarcely attracts notice. The detection of these systems relatively close to us is an important branch of study; they are distinguished by the members having all precisely equal and parallel motions. The stars seem to be at quite ordinary stellar distances apart, and their mutual attraction is too weak to cause any appreciable orbital motion. They are not held together by any force; and we can only infer that they continue to move together because no force has ever intervened to separate them.

These "moving clusters" are contained within the central aggregation of stars. Many of the globular clusters, though much more distant, are probably also contained in it; others, however, may be situated in the star-clouds of the Milky Way. Their distribution in the sky is curiously uneven; they are nearly all contained in one hemisphere. They are most abundant in Sagittarius and Ophiuchus,

near a brilliant patch of the Milky Way, which is undoubtedly the most extraordinary region in the sky. This might be described as the home of the globular clusters.

We shall also have to consider the nebulæ, and their relation to the system of the stars. At this stage it may be sufficient to state that under the name "nebulæ" are grouped together a number of objects of widely differing constitution; we must not be deceived into supposing that the different species have anything in common. There is some reason for thinking that the spiral or "white" nebulæ are objects actually outside the whole stellar system, that they are indeed stellar systems coequal with our own, and isolated from us by a vast intervening void. But the gaseous irregular nebulæ, and probably also the planetary nebulæ, are more closely associated with the stars and must be placed among them.

It is now time to turn from this outline of the leading phenomena to a more detailed consideration of the problems. The procedure will be to treat first the nearest stars, of which our knowledge is unusually full and direct. From these we pass to other groups that happen to be specially instructive. From this very limited number of stars, a certain amount of generalisation is permissible, but our next duty is to consider the motions of the stars in general; this will occupy Chapters V.—VII. After considering the dependence of the various phenomena on spectral type, we pass on to the problems of stellar distribution. This comes *after* our treatment of stellar motions, because the proper motions are, when carefully treated, among the most important sources of information as to the distances of stars. In Chapter XI. we pass to subjects—the Milky Way and Nebulæ—of which our knowledge is even more indefinite. The concluding Chapter attempts to introduce the problem of the dynamical forces under which motions of the stellar system are maintained.

CHAPTER III

THE NEAREST STARS

Most of our knowledge of the distribution of the stars is derived by indirect methods. Statistics of stellar magnitudes and motions are analysed and inferences are drawn from them. In this Chapter, however, we shall consider what may be learnt from those stars the distances of which have been measured directly; and, although but a small sample of the stellar system comes under review in this way, it forms an excellent starting point, from which we may proceed to investigations that are usually more hypothetical in their basis.

For a parallax-determination of the highest order of accuracy, the probable error is usually about $0''\cdot01$. Thus the position of a star in space is subject to a comparatively large uncertainty, unless its parallax amounts to at least a tenth of a second of arc. How small a ratio such stars bear to the whole number may be judged from the fact that the median parallax of the stars visible to the naked eye is only $0''\cdot008$; as many naked-eye stars have parallaxes below this figure as above it. We are therefore in this Chapter confined to the merest fringe of the surrounding universe, making for the present no attempt to penetrate into the general mass of the stars.

Table 3 shows all the stars that have been found to have

parallaxes of 0″·20 or greater.* Only the most trustworthy determinations have been accepted, and in most cases at least two independent investigators have confirmed one another. The list is based mainly on Kapteyn and Weersma's compilation.[1]

TABLE 3.

The Nineteen Nearest Stars.

(Stars distant less than five parsecs from the Sun.)

Star.	Magnitude.	Spectrum.	Parallax.	Luminosity (Sun = 1).	Remarks.
			″		
Groombridge 34	8·2	Ma	0·28	0·010	Binary
η Cassiopeiae	3·6	F8	0·20	1·4	Binary
τ Ceti	3·6	K	0·33	0·50	
ε Eridani	3·3	K	0·31	0·79	
CZ 5^h 243	8·3	G–K	0·32	0·007	
Sirius	−1·6	A	0·38	48·0	Binary
Procyon	0·5	F5	0·32	9·7	Binary
Lal. 21185	7·6	Ma	0·40	0·009	
Lal. 21258	8·9	Ma	0·20	0·011	
OA (N.) 11677	9·2	—	0·20	0·008	
α Centauri	0·3	G, K5	0·76	{2·0, 0·6}	Binary
OA (N.) 17415	9·3	F	0·27	0·004	
Pos. Med. 2164	8·8	K	0·29	0·006	Binary
σ Draconis	4·8	K	0·20	0·5	
α Aquilae	0·9	A5	0·24	12·3	
61 Cygni	5·6	K5	0·31	0·10	Binary
ε Indi	4·7	K5	0·28	0·25	
Krüger 60	9·2	—	0·26	0·005	Binary
Lacaille 9352	7·4	Ma	0·29	0·019	

It is of much interest to inquire how far this list of nineteen stars is exhaustive. Does it include all the stars in a sphere about the Sun as centre with radius five parsecs? In one respect the Table is admittedly incomplete; for stars fainter than magnitude 9·5 (on the B.D. scale), determinations are entirely lacking. A $9^m·5$ star with a parallax 0″·2 would have a luminosity 0·006, the

* In using a table of this kind I am following the Astronomer Royal, F. W. Dyson, who first showed me its importance.

Sun being the unit; so that, in general, stars giving less than 1/200th of the light of the Sun could not be included in the list. The distribution of the luminosities in column five of the Table leads us to expect that these very feeble stars may be rather numerous.

Admitting, then, that Table 3 breaks off at about luminosity 0·006, and that in all probability numerous fainter stars exist within the sphere, how far is it complete above this limit? Generally speaking, stars are selected for parallax-determinations on account of their large proper motions. Most of the very bright stars have also been measured, but in no case has a parallax greater than 0″·2 been found which was not already rendered probable by the existence of a large proper motion. To form some idea of the completeness with which the stars have been surveyed for parallax, consider those stars the motions of which exceed 1″ per annum. F. W. Dyson has given a list of ninety-five of these stars,[2] and it is probable that his list is nearly complete, at least as far as the ninth magnitude; the Durchmusterungs and Meridian Catalogues of most parts of the sky have involved so thorough a scrutiny, that it would be difficult for motions as large as this to remain unnoticed. Of these ninety-five stars, sixty-five may be considered to have well-determined parallaxes, or at least the determinations have sufficed to show that they lie beyond the limits of our sphere; among the former are seventeen of the nineteen stars of Table 3. For the remaining thirty, either measurements have been unattempted, or the determinations do not negative the possibility of their falling within the sphere. This remainder is not likely to be so rich in large parallaxes, because it includes a rather large proportion of stars which only just exceed the annual motion of 1″; but there are some notable exceptions. The star Cordoba 32416, mag. 8·5, having the enormous annual motion of 6″·07 seems to have been left alone entirely. It may be expected that

further examination of these thirty stars will yield four or five additional members for our Table.

Of stars with annual proper motions less than 1″ the Table contains only two. It is not difficult to show that this is an inadequate proportion. The median parallax of stars distributed uniformly through the sphere (of radius 5 parsecs) is 0″·25 ; now for a star of that parallax an annual motion of 1″ would be equivalent to a linear transverse motion of 20 km. per sec. Approximately then for our sphere

$$\frac{\text{No. of } PM\text{'s} > 1''}{\text{No. of } PM\text{'s} < 1''} = \frac{\text{No. of transverse motions} > 20 \text{ km./sec.}}{\text{No. of transverse motions} < 20 \text{ km./sec.}}$$

Now our general knowledge of stellar velocities, derived from other sources, is probably sufficiently good to give a rough idea of the latter ratio ; for it may be expected that the distribution of linear velocities within the sphere will not differ much from the general distribution outside. Taking the average radial velocity of a star as 17 km. per sec. (the figure given by Campbell for types K and M, which constitute the great majority of the stars), a Maxwellian distribution would give 64 per cent. of the transverse motions greater than 20 km. per sec. and 36 per cent. less—a ratio of 1·8 : 1. The addition of the solar motion will increase the proportion of high velocities ; and in those parts of the sky where it has its full effect the ratio is nearly 3 : 1. Probably we shall not be far wrong in assuming that there will be two-fifths as many stars with motions below 1″ per annum as above it.

Having regard to these considerations, the calculation stands thus—

No. of stars in the Table with proper motion greater than 1″ .	17
Proportionate allowance for stars not yet examined	5
Due proportion with proper motion less than 1″, say	9
The Sun	1
Total	32

To this must be added an unknown but probably considerable number of stars the luminosity of which is less than 1/200th of the Sun.

In round numbers we shall take thirty as the density of the stars within the sphere (tacitly ignoring the intrinsically faint stars). As twenty of these are actually identified, the number may be considered to rest on observation with very little assistance from hypothetical considerations.

This short list of the nearest stars well repays a careful study. Many of the leading facts of stellar distribution are contained in it; and, although it would be unsafe to generalise from so small a sample, results are suggested that may be verified by more extensive studies.

Perhaps the most striking feature is the number of double stars. It will be seen that eight out of the nineteen are marked "binary." Why some stars have split into two components, whilst others have held together, is an interesting question; but it appears that the fission of a star is by no means an abnormal fate. The stars which separate into two appear to be not much less numerous than those which remain intact. The large number of discoveries of variable radial velocity made with the spectroscope confirms this inference, though the spectroscopists generally do not give quite so high a proportion. W. W. Campbell[3] from an examination of 1600 stars concludes that one-quarter are spectroscopic binaries. But this proportion must be increased if visual binaries are included (for these are not usually revealed by the spectroscope); and, in addition, there must be pairs too far separated to be detected as spectroscopic binaries, but too distant from us to be recognised visually. E. B. Frost, examining the stars of Type B, found that two-fifths of those on his programme were binary; he also found that in Boss's Taurus cluster the proportion was one-half.[4]

In the Ursa Major cluster nine stars out of fifteen are known to be binary.[5] Apparently the division into two bodies takes place at a very early stage in a star's history or in the pre-stellar state, as is evidenced by the high proportion found in the earliest spectral type. As time passes the components separate further from each other and the orbital velocity becomes small, so that in the later types an increasing proportion escapes detection. There thus seems no reason to doubt that the proportion eight out of nineteen very fairly represents the general average, but at the very lowest it cannot be less than one in three.

The luminosities of the stars in the Table range from 48 to 0·004, that of the Sun being taken as unit. We have already seen that the lower limit is due to the fact that our information breaks off near this point, and it is natural to expect that there must be a continuous series of fainter bodies terminating with totally extinct stars. At the other end of the scale a larger sample would undoubtedly contain stars of much greater luminosity; but these are comparatively rare in space. Arcturus, for example, is from 150 to 350 times as bright as the Sun, Antares at least 180 times, whilst Rigel and Canopus can scarcely be less than 2000 times as bright, allowing in each case a wide margin for the possible uncertainty of the measured parallaxes. There is little doubt that these estimates err on the side of excessive caution.

For a star of the same intrinsic luminosity as the Sun to appear as bright as the sixth magnitude, its parallax must be not less than 0″·08. Since there is no doubt that the majority of stars visible to the naked eye are much further away than this, it follows that the great majority of these stars must be much brighter, in fact more than a hundred times as bright as the Sun. We might hastily suppose that the Sun is therefore far below the average brilliancy. But Table 3 reveals a very different state of

things. Of the nineteen stars, only five exceed it, whilst fourteen are fainter. The apparent paradox directs attention to a fact which we shall have occasion to notice frequently. The stars visible to the naked eye, and the stars enumerated in the catalogues, are quite unrepresentative of the stars as a whole. The more intensely luminous stars are seen and recorded in numbers out of all proportion to their actual abundance in space. It is of great importance to bear in mind this limitation of statistical work on star-catalogues; we ought to consider whether the results derived from the very special kind of stars that appear in them may legitimately be extended to the stars as a whole.

In the same way, the Table gives a very different idea of the proportions in which the different types of spectra occur from the impression we should gather by examining the catalogues. Four stars of Type M are included, although in the catalogues this class forms only about a fifteenth of the whole number. On the other hand, Type B stars (the Orion type), which are rather more numerous than Type M in the catalogues, have not a single representative here. The explanation is that these M stars are usually very feebly luminous objects (as may be seen in the Table), and can rarely be seen except in our immediate neighbourhood. Type B stars on the other hand are intensely bright, and although they occur but sparsely in space, we can record even those near the limits of the stellar system, making a disproportionately large number. Again in the catalogues the Sirian Type stars (A) and the Solar Type (F, G, K) are about equally numerous, but in this definite volume of space the latter outnumber the former by ten to two.

In order to obtain more extensive data as to the true proportions of the spectral types, and the relation of spectral type to luminosity, Table 4 has been drawn up, containing the stars with fairly well-determined parallaxes

between 0″·19 and 0″·11. These are not generally so trustworthy as the parallaxes of Table 3, because chief attention has naturally been lavished on those stars known to be nearest to us; but the standard is fairly high, a great many measured parallaxes being rejected as too uncertain. Those marked with an asterisk are the best determined and may be considered equal in accuracy to the parallaxes of Table 3; but as the parallaxes are smaller, the proportionate uncertainty of the calculated luminosity is greater.

TABLE 4.

Stars distant between 5 and 10 parsecs from the Sun.

Star.	Magnitude.	Spectrum.	Annual Proper Motion.	Parallax.	Luminosity (Sun = 1).
			″	″	
ζ Toucanæ	4·3	F8	2·07	0·15	1·3
*β Hydri	2·9	G	2·24	0·14	5·4
54 Piscium	6·1	K	0·59	0·15	0·26
Mayer 20	5·8	K	1·34	0·16	0·28
*μ Cassiopeiæ	5·3	G5	3·75	0·11	1·0
δ Trianguli	5·1	G	1·16	0·12	1·0
Pi. 2h 123	5·9	G5	2·31	0·14	0·33
e Eridani	4·3	G5	3·15	0·16	1·15
*δ Eridani	3·3	K	0·75	0·19	2·1
o^2 Eridani	4·5	G5	4·08	0·17	0·84
λ Aurigæ	4·8	G	0·85	0·11	1·5
*Weisse 5h 592	8·9	Ma	2·23	0·18	0·013
Pi. 5h 146	6·4	G2	0·55	0·11	0·32
*Fed. 1457–8	7·9	Ma	1·69	0·16	0·042
*Groombridge 1618.	6·8	K	1·45	0·18	0·09
43 Comæ	4·3	G	1·18	0·12	2·2
*Lalande 25372	8·7	K	2·33	0·18	0·017
Lalande 26196	7·6	G5	0·68	0·14	0·074
*Pi. 14h 212	5·8	K	2·07	0·17	0·26
*Groningen VII., No. 20	10·7	—	1·22	0·13	0·005
ζ Herculis	3·0	G	0·61	0·14	5·0
*Weisse 17h 322	7·8	Ma	1·36	0·12	0·08
70 Ophiuchi	4·3	K	1·15	0·17	1·1
*17 Lyrae C.	11·3	—	1·75	0·13	0·003
Fomalhaut	1·3	A3	0·37	0·14	25·0
*Bradley 3077	5·6	K	2·11	0·14	0·45
*Lalande 46650	8·9	Ma	1·40	0·18	0·013

Table 4 is far from being a complete list of the stars within the limits. There should be about 200 stars in

this volume of space, but only 27 are given here. However, the incompleteness will not much affect the present inquiry, except that a number of the faintest stars, which are especially of Types K and Ma, will naturally be lost. This is noticeable when the list is compared with Table 3.

Collecting the stars of different spectral types, we have the following distribution of luminosities from Tables 3 and 4:

Type of Spectrum.	Luminosities. Parallaxes greater than 0·20″.	Luminosities. Parallaxes from 0·19″ to 0·11″.
A . . .	48·0	——
A3 . .	——	25·0
A5 . .	12·3	——
F . . .	0·004	——
F5 . .	9·7	——
F8 . .	1·4	1·3
G . . .	2·0	*5·4, 5·0, 2·2, 1·5, 1·0
G2 . .	——	0·32
G5 . .	——	1·15, *1·0, 0·84, 0·33, 0·074
K . .	0·79, 0·5, 0·5, 0·006	*2·1, 1·1, *0·45, 0·28, *0·26 0·26, *0·09, *0·017
K5 . .	0·6, 0·25	——
Ma . .	0·019, 0·011 0·010, 0·009	*0·08, *0·042, *0·013, *0·013

This summary shows a remarkable tendency towards equality of brightness among stars of the same type, and there is a striking progressive diminution of brightness with advance in the stage of evolution. The single star of Type F (strictly F0) makes a curious exception. This star, OA (N) 17415, was measured with the heliometer by Krüger as early as 1863; apparently the determination was an excellent one. It would perhaps be desirable to check his result by observations according to more modern methods; but we are inclined to believe that the exception is real.

It would be tempting to conclude that the great range in absolute luminosity of the stars is mainly due to differences of type, and that within the same spectral class

* Well-determined parallaxes.

the range is very limited. We might apply this in searching for large parallaxes; for it would seem that, since stars of Type Ma have so far been found to be of very feeble luminosity, any star of that class which appears bright must be very close to us. This hope is not fulfilled. The bright Ma stars which have been measured—Betelgeuse, Antares, η Geminorum and δ Virginis—have all very small parallaxes, and are certainly much more luminous even than Sirius, the brightest star in the Tables. It is perhaps unfortunate that these brilliant exceptions force themselves on our notice, whilst the far greater number of normal members of the class are too faint to attract attention; we must not be led into an exaggerated idea of the number of these luminous third type stars. But it is clear that, notwithstanding the tendency to equality, if a large enough sample is taken, the range of luminosity is very great indeed. We shall have to return to this subject in Chapter VIII.

Table 5 contains some details as to the motions of the nineteen nearest stars. The transverse velocities are formed by using the measured parallaxes to convert proper motions into linear measure.* The radial motions from spectroscopic observations are added when available. These motions are relative to the Sun; if we wished to refer them to the centroid of the stars, we should have to apply the solar motion of 20 km. per sec., which might either increase or decrease the velocities according to circumstances, but would usually somewhat decrease them. After allowing for this, the commonness of large velocities is still a striking and most surprising feature. The ordinary studies of stellar motions do not lead us to expect anything of the kind; and in fact it is not easy to reconcile the general

* The formula, which is often useful, is

$$\text{Linear speed} = \frac{\text{annual proper motion}}{\text{parallax}} \times 4{\cdot}74 \text{ km. per sec.}$$

investigations with the results of this special study of a small collection of stars. Taking the fastest moving stars, Type M, Campbell found an average radial motion of 17 km. per sec. Assuming a Maxwellian distribution of velocities, this would give for a transverse motion (*i.e.*, motion in two dimensions)

Speed greater than	60 km. per sec.	1 star in 53
,, ,, ,,	80 ,,	1 star in 1,100
,, ,, ,,	100 ,,	1 star in 60,000

The presence of 3 stars in the list with transverse velocities of more than 100 km. per sec. (and therefore certainly more than 80 km. per sec., when the solar motion is removed) is wholly at variance with the above statistical scheme.

TABLE 5.

Motions of the nineteen nearest stars.

Star.	Proper Motion.		Radial Velocity.	Stream.
	Arc.	Linear.		
	"	km. per sec.	km. per sec.	
Groombridge 34	2·85	48	—	I.
η Cassiopeiae	1·25	30	+10	I.
τ Ceti	1·93	28	−16	II.
ϵ Eridani	1·00	15	+16	II.
CZ 5^h 243	8·70	129	+242	II.
Sirius	1·32	16	−7	II.
Procyon	1·25	19	−3	I. ?
Lalande 21185	4·77	57	—	II.
Lalande 21258	4·46	106	—	I.
OA (N) 11677	3·03	72	—	I.
α Centauri	3·66	23	−22	I.
OA (N) 17415	1·31	23	—	II.
Pos. Med. 2164	2·28	37	—	I.
σ Draconis	1·84	43	+25	II.
α Aquilae	0·65	13	−33	I.
61 Cygni	5·25	80	−62	I.
ϵ Indi	4·67	79	−39	I.
Krüger 60	0·92	17	—	II.
Lacaille 9352	7·02	115	+12	I.

We cannot attribute the result to errors in the parallaxes, for if it should happen that the parallax has

been overestimated, the speed will have been underestimated; and it is scarcely likely that any of these stars have parallaxes appreciably greater than those assigned in the Table. A possible criticism is that these stars have been specially selected for parallax-measurement, because they were known to have large proper motions; but the objection has not much weight unless it is seriously suggested that there are, in this small volume, the hundreds or even thousands of stars of small linear motion that the statistical scheme seems to require. Moreover we have already shown that on the ordinary view as to stellar motions, seven additional stars would supply the loss due to the neglect of stars with motions less than 1″ per annum.

Nor can we help matters by throwing over the Maxwellian law. Originally used as a pure assumption, this law has been confirmed in the main by recent study of the radial motions. We should, however, be prepared to admit that it may not give quite sufficient very large motions. It has long been known that certain stars, as Arcturus and Groombridge 1830, had excessive speeds that seemed to stand outside the ordinary laws. But we find that the average transverse motion of the nineteen stars is fifty km. per sec.; this considerably exceeds the average speed we should deduce from the stars that come into the ordinary investigations. In round numbers, a mean speed of thirty km. per sec. relative to the Sun would have been expected.*

Thus once more it is found that the survey of stars in the limited volume of space very near to the Sun leads to results differing from those derived from the stars of the catalogues. We may fall back on the same explana-

* An average radial speed of 17 km. per sec. gives an average transverse speed of 26·5 km. per sec., the factor being $\frac{\pi}{2}$ whatever the law (Maxwellian or otherwise) of stellar motions. This does not include the solar motion, which, however, would not increase the result very greatly.

tion as before, that the catalogues give a very untypical selection of the stars. But this time the result is more surprising; it would scarcely have been expected that the catalogue-selection, which is purely by brightness, would have so large an effect on the motions. Yet this seems to be the case. We notice that the three stars with transverse speeds of more than 100 km. per sec. have luminosities 0·007, 0·011 and 0·019. These would be far too faint to come into the ordinary statistical investigations. The five stars brighter than the Sun have all very moderate speeds. Setting the nine most luminous stars against the ten feeblest, we have—

	Luminosity.	Mean transverse speed.
9 Brightest stars . .	48·0 to 0·25	29 km. per sec.
10 Faintest ,, . .	0·10 ,, 0·004	68 ,,

It is the stars with luminosity less than 1/10th that of the Sun, with which we are scarcely ever concerned in the ordinary researches on stellar motions, that are wholly responsible for the anomaly. The nine bright stars simply confirm our general estimate of thirty km. per sec. for the average speed.

The stars of Table 4 also add a little evidence pointing in the same direction. The parallaxes are scarcely accurate enough (proportionately to their size) to be used for this purpose; but we give the result for what it is worth. It may be noted that the parallaxes are likely to be a little overestimated and therefore both luminosities and speeds will be underestimated. On the other hand, the influence of selection (on account of large proper motion) will be greater than in Table 3, tending to increase the mean speed unduly. There are nine stars given in Table 4 as having luminosities less than 0·1; their mean speed is forty-eight km. per sec. Thus these stars have speeds considerably in excess of the thirty km. per sec. originally expected. They do not, however, include any excessive speeds.

It would be very desirable to have more evidence on this point before drawing a general conclusion; but our task is to sum up the present state of our knowledge, however fragmentary. The stars, of which the proper motions and radial velocities are ordinarily discussed, are almost exclusively those at least as bright as the Sun. It is generally tacitly assumed that the motions of the far more numerous stars of less brilliance will be similar to them. But the present discussion affords a strong suspicion that there exists a class of stars, comprising the majority of those having a luminosity below 0·1, the speeds of which are on the average twice as great as the fastest class ordinarily considered. Apparently the progressive increase of velocity with spectral type does not end with the seventeen km. per sec. of the brilliant members of Type M, but continues for fainter stars up to at least twice that speed.

In the final column of Table 5, each star has been assigned to its respective star-stream according to the direction in which it is moving. We see that eleven stars probably belong to Stream I, and eight probably to Stream II. This is in excellent accordance with the ratio 3 : 2 derived from the discussion of the 6000 stars of Boss's Catalogue. We are not able to detect any significant difference between the luminosities, spectra, or speeds of the stars constituting the two streams. The thorough inter-penetration of the two star-streams is well illustrated, since we find even in this small volume of space that members of both streams are mingled together in just about the average proportion.

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4. **Frost, *Astrophysical Journal*, Vol. 29, p. 237.**
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CHAPTER IV

MOVING CLUSTERS

THE investigation of stellar motions has revealed a number of groups of stars in which the individual members have equal and parallel velocities. The stars which form these associations are not exceptionally near to one another, and indeed it often happens that other stars, not belonging to the group, are actually interspersed between them. We may perhaps arrive at a better understanding of these systems by recalling a few elementary considerations regarding double stars.

In only a small proportion of the double stars classed as "physically connected" pairs, has the orbital motion of one component round the other been detected. In most cases the connection is inferred from the fact that the two stars are moving across the sky with the same proper motion in the same direction. The argument is that, apart from exceptional coincidences, the equality of angular motion signifies both an equality of distance and an equality of linear velocity. Accordingly the two stars must be close together in space, and their motions are such that they must have remained close together for a long period. Having established the fact that they are permanent neighbours, we may rightly deduce that their mutual gravitation will involve some orbital motion, though it may be too slow to detect; but that is a subsidiary

matter, and, in speaking of physical connection, we are not thinking of two stars tied together by an attractive force. The connection, if we try to interpret it, appears to be one of origin. The components have originated in the same part of space, probably from a single star or nebula; they started with the same motion, and have shared all the accidents of the journey together. If the path of one is being slowly deflected by the resultant pull of the stellar system, the path of the other is being deflected at the same rate, so that equality of motion is preserved. It is true that the mutual attraction in these widely separated binaries may help to prevent the stars separating; but it is a very feeble tie, and, in the main, community of motion persists because there are no forces tending to destroy it.

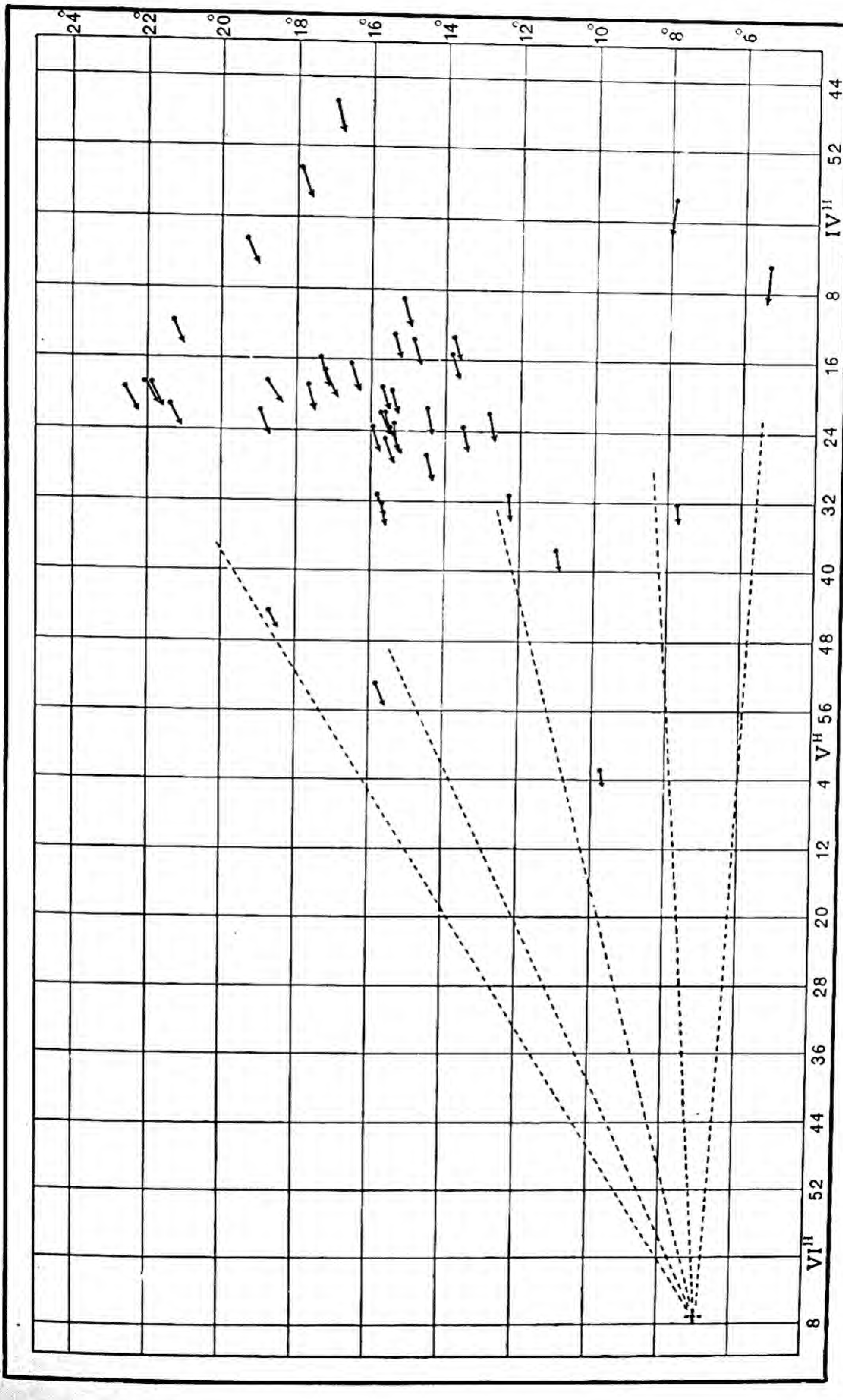
From this point of view we may have physically connected pairs separated by much greater and even by ordinary stellar distances, remembering, however, that the greater the distance the more likely are they to lose their common velocity by being exposed to different forces. It is known that exceedingly wide pairs do exist. The case of A Ophiuchi and Bradley 2179 may be instanced; these stars are separated by about 14′, but have the same unusually large motion of 1″·24 per annum in the same direction. In general it would be difficult to detect pairs of this kind; for unless the motion is in some way remarkable, an accidental equality of motion must often be expected, and it would be impossible to distinguish the true pairs from the spurious. It is only when there is something unusual in the amount or in the direction of the motion that there are grounds for believing the equality is not accidental.

In the Moving Clusters we find a closely similar kind of physical connection. They are considerable groups of stars, widely separated in the sky, but betraying their association by the equality of their motions. The most

thoroughly investigated example of a moving cluster is the Taurus-stream, which comprises part of the stars of the Hyades and other neighbouring stars. The existence of a great number of stars with associated motions in this region was pointed out by R. A. Proctor; but the researches of L. Boss have shown the nature of the connection in a new light. Thirty-nine stars are recognised as belonging to the group, distributed over an area of the sky about 15° square; there can be no doubt that many additional fainter stars [1] in the region also belong to the cluster but, until better determinations of their motions have been made, these cannot be picked out with certainty.

The first criterion in such a case as this is that the motions should all appear to converge towards a single point in the sky; in Fig. 2 the arrows indicate the observed motions of these stars, and the convergence is well shown. From this it may be deduced that the motions are parallel; for lines parallel in space appear, when projected on a sphere, to converge to a point. It is true that the same appearance would be produced if the motions were all converging to or diverging from a point, but either supposition is obviously improbable. Theoretically there may be a slight degree of divergence, the cluster having been originally more compact; but calculation shows that on any reasonable assumption as to the age of the cluster the divergence must be quite negligible. As the stars are not all at the same distance from the Sun, the fact that the speeds are all equal cannot be demonstrated in the same exact way; but, allowing for the foreshortening of the apparent motion in the front of the cluster as compared with the rear, the proper motions all agree with one another very nearly. The divergences are just what we should expect if the cluster extends towards the Sun and away from it to the same distance that it extends laterally.

It is clear that in a cluster of this kind the equality and



Emery Walker Ltd. sc.

FIG. 2.—Moving Cluster in Taurus. (Boss.)

parallelism of the motions must be extremely accurate, otherwise the cluster could not have held together. Suppose that the motion of one member deviated from the mean by one km. per sec.; it would draw away from the rest of the cluster at the rate of one astronomical unit in $4\frac{3}{4}$ years. In ten million years it would have receded ten parsecs (the distance corresponding to a parallax of $0''\cdot10$). We shall see later that the actual dimensions of the cluster are not so great as this; the remotest member is about seven parsecs from the centre. According to present ideas, ten million years is a short period in the life even of a planet like the earth; the age of the Taurus-cluster, which contains stars of a fairly advanced type of evolution, must be vastly greater than this. From the fact that it still remains a compact group we deduce that the individual velocities must all agree to *within a small fraction of a kilometre per second.*

The very close convergence of the directions of motion of these stars supports this view; the deviations may all be attributed to the accidental errors of observation. In fact the mean deviation (calculated − observed) in position angle is $\pm 1^\circ\cdot 8$, whereas the expected deviation, due to the probable errors of the observed proper motions, is greater than this,—a paradox which is explained by the fact that stars for which the accidental error is especially great would not be picked out as belonging to the group.

To complete our knowledge of this cluster, one other fact of observation is required; namely, the motion in the line of sight of any one of the individual stars. Actually six have been measured, and the results are in satisfactory accordance. The data are now sufficient to locate completely not only the cluster but its individual members and also to determine the linear motion, which, as has been shown, must be the same for all the stars to a very close approximation. This can be done as follows:—

The position of the convergent point, shown at the extreme left of Fig. 2, is found to be

R.A. $6^h\ 7^m{\cdot}2$ Dec. $+6°\ 56'$ (1875·0)

with a probable error of $\pm 1°{\cdot}5$ chiefly in right ascension. If O is the observer (Fig. 3), let OA be the direction of this convergent point.

Consider one of the stars S of the cluster. Its motion * in space ST must be parallel to OA. Resolve ST into transverse and radial components SX and SY. If SY has been measured by the spectroscope, we can at once find ST for

$$\begin{aligned} ST &= SY \sec TSY \\ &= SY \sec AOS \end{aligned}$$

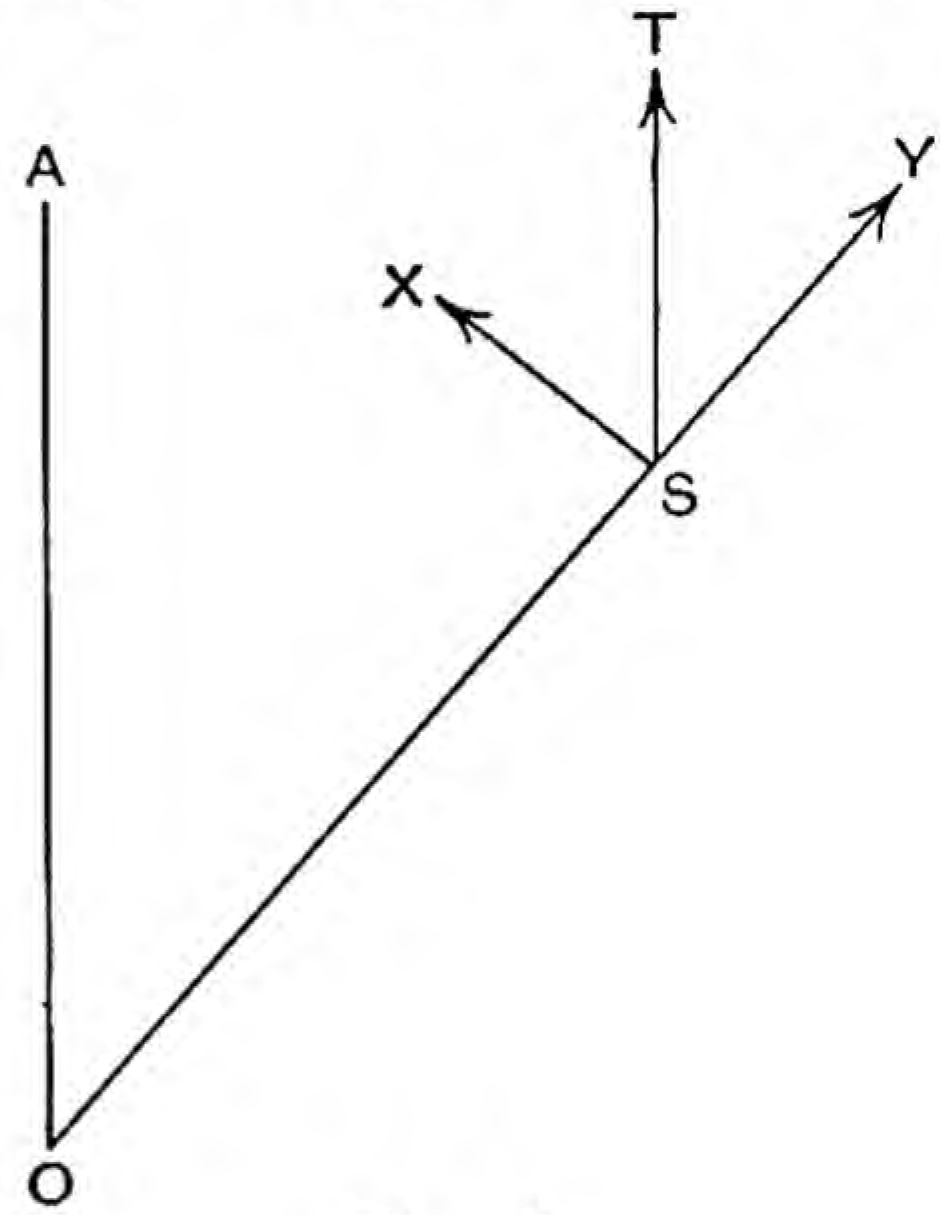


FIG. 3.

and, since A and S are known points on the celestial sphere, the angle AOS is known.

Since the velocity ST is the same for every star of the cluster, it is sufficient to determine it from any one member the radial velocity of which has been measured. The result is found to be 45·6 km. per sec.

We next find the transverse velocity for each star, which is equal to

45·6 sin AOS km. per sec.

And the stars' distances are given by

transverse velocity = distance × observed proper motion

when these are expressed in consistent units.

It will be seen that the distance of every star is found by this method, not merely those of which the radial velocity is known. The distance is found with a percentage accuracy

* The motions considered here are all measured relatively to the Sun.

about equal to that of the observed proper motion; for the other quantities which enter into the formulae are very well-determined. As the proper motions of these stars are large and of fair accuracy, the resulting distances are among the most exactly known of any in the heavens. The parallaxes range from 0″·021 to 0″·031, with a mean of 0″·025. A direct determination by photography, the result of the cooperation of A. S. Donner, F. Küstner, J. C. Kapteyn, and W. de Sitter,[2] has yielded the mean value 0″·023±0″·0025, which, though presumably less accurate than the result obtained indirectly, is a satisfactory confirmation of the legitimacy of Boss's argument.

From these researches the Taurus-cluster appears to be a globular cluster with a slight central condensation; its whole diameter is rather more than ten parsecs. The question arises whether this system can be regarded as similar to the recognised globular clusters revealed by the telescope. If there were no more members than the thirty-nine at present known, the closeness of arrangement of the stars would not be greater than that which we have found in the immediate neighbourhood of the Sun. But it appears that the 39 are all much brighter bodies than the Sun, and it is not fair to make a comparison with the feebly luminous stars discussed in the last chapter. According to a rough calculation the members of the Taurus-cluster may be classified as follows:

5	stars with luminosity	5	to	10	times that of the Sun
18	,,	10	,,	20	,, ,, ,,
11	,,	20	,,	50	,, ,, ,,
5	,,	50	,,	100	,, ,, ,,

In the vicinity of the Sun we have nothing to compare with this collection of magnificent orbs. These stars, it is true, are separated by distances of the usual order of magnitude; but their exceptional brilliancy marks out this portion of space from an ordinary region. Whether

there are or are not other fainter members accompanying them, the term cluster is appropriate enough. There can be no doubt that, viewed from a sufficient distance, this assemblage would have the general appearance of a globular star-cluster.

The known motion of the Taurus-cluster permits us to trace its past and future history. It was in perihelion 800,000 years ago; the distance was then about half what it is now. Boss has computed that in 65,000,000 years it will (if the motion is undisturbed) appear as an ordinary globular cluster 20′ in diameter, consisting largely of stars from the ninth to the twelfth magnitude.

It is interesting to note that a cluster of the size of this Taurus group must contain many interloping stars not belonging to it. Even if we omit the outlying members, the system fills a space equal to a sphere of at least 5 parsecs radius. Now such a sphere in the neighbourhood of the Sun contains about 30 stars. We cannot suppose that a vacant lane among the stars has been specially left for the passage of the cluster. Presumably then the stars that would ordinarily occupy that space are actually there —non-cluster stars interspersed among the actual members of the moving cluster. It is a significant fact that the penetration of the cluster by unassociated stars has not disturbed the parallelism of the motions or dispersed the members.

The Ursa Major system is another moving cluster of which detailed knowledge has been ascertained. It has long been known that five stars of the Plough, viz., β, γ, δ, ϵ and ζ Ursae Majoris, form a connected system. By the work of Ejnar Hertzsprung it has been shown that a number of other stars, scattered over a great part of the sky, belong to the same association. The most interesting of these scattered members is Sirius; and for it the evidence of the association is very strong. Its parallax and radial velocity are both well-determined, and agree

with the values calculated from the motion of the whole cluster. The method by which the common velocity is found, and the individual stars are located in space, is the same as that employed for the Taurus-cluster. The velocity is 18·4 km. per sec. towards the convergent point R.A. 127°·8, Dec. + 40°·2, when measured relatively to the Sun. When the solar motion is allowed for, the "absolute" motion is 28·8 km. per sec. towards R.A. 285°, Dec. –2°. As this point is only 5° from the galactic plane, the motion is approximately parallel to the galaxy.

In Table 6 particulars of the individual stars are set down, including the parallaxes and radial velocities deduced by Hertzsprung from the known motion of the system. In most cases the calculated radial velocities have been confirmed by observation;[3] but for nearly all the parallaxes, it has not been possible as yet to test the values given. It is quite likely that one or more stars have been wrongly included; but there can be little doubt that the majority are genuine members of the group. The rectangular co-ordinates are given in the usual unit (the parsec), the Sun being at the origin, Oz directed towards the convergent point R.A. 127°·8, Dec. + 40°·2, and Ox towards R.A. 307°·8, Dec. + 49°·8, so that the plane zOx contains the Pole. If a model of the system is made from these data, it is found, as H. H. Turner[4] has shown, that the cluster is in the form of a disk; its plane being nearly perpendicular to the galactic plane. The flatness is very remarkable, the average deviation of the individual stars above or below the plane being 2·0 parsecs, a distance small in comparison with the lateral extent of the cluster, viz., 30 to 50 parsecs. In the last column are given the absolute luminosities in terms of the Sun as unit; it is interesting to note that the three stars of Type F are the faintest, with luminosities 10, 9, and 7 respectively.

TABLE 6.

The Ursa Major System.

Star.	Mag.	Spectrum.	Computed. Parallax.	Computed. Radial Velocity.	Rectangular Co-ordinates. x	y	z	Luminosity (Sun=1).
			"	km. sec.				
β Eridani . . .	2·92	A2	0·034	−7·5	−13·8	−23·3	12·1	96
β Aurigæ . . .	2·07	Ap	0·024	−16·0	7·8	−18·8	36·3	410
Sirius	−1·58	A	0·387	−8·5	−2·0	−1·1	1·2	46
37 Ursæ Maj. .	5·16	F	0·045	−16·6	7·6	5·8	19·9	7
β Ursæ Maj. .	2·44	A	0·047	−16·1	7·6	6·9	18·7	76
δ Leonis . . .	2·58	A2	0·084	−14·4	−2·3	7·1	9·3	21
γ Ursæ Maj. .	2·54	A	0·042	−15·0	8·9	10·5	19·3	87
δ Ursæ Maj. . .	3·44	A2	0·045	−14·4	9·8	9·7	17·2	32
Groom. 1930 .	5·87	F	0·028	−13·4	18·6	15·1	25·7	9
ϵ Ursæ Maj. .	1·68	Ap	0·042	−13·2	11·4	11·7	16·9	190
78 Ursæ Maj. .	4·89	F	0·042	−13·0	12·0	11·8	16·8	10
ζ Ursæ Maj. . .	{ 2·40 / 3·96 }	{ Ap / A2 }	0·043	−12·2	11·9	12·5	15·3	{ 93 / 22 }
α Coronæ . . .	2·31	A	0·041	−2·2	12·0	20·9	2·9	110

The stars of the Orion type of spectrum present several examples of moving clusters. In the Pleiades we have an evident cluster, in the ordinary sense of the term, and, as might be expected, the motions of the principal stars and at least fifty fainter stars are equal and parallel.* The bright stars of the constellation Orion itself (with the exception of Betelgeuse, the spectrum of which is not of Type B) also appear to form a system of this sort, the evidence in this case being mainly derived from their radial velocities, since the transverse motions are all exceedingly small. In Orion a faint nebulosity forming an extension of the Great Orion Nebula, has been discovered, which appears to fill the whole region occupied by the stars; it probably consists of the lighter gases and other materials not yet absorbed by the stars which are developing. The velocity of the nebula in the line of sight agrees with that of the

* This is the case as regards the proper motions. The radial velocities of the six brightest stars show some surprising differences (Adams, *Astrophysical Journal*, Vol. 19, p. 338), but owing to the difficult nature of the spectra the determinations are not very trustworthy.

stars of the constellation. A similar nebulosity is found in the Pleiades.

In the case of these, the youngest of the stars, the argument by which we deduced the accurate equality of motion in the Taurus-cluster scarcely applies; particularly in Orion, the dimensions of which must be at least a hundred times greater than those of the Taurus-cluster, it is just possible that the associated stars may be dispersing rather rapidly.

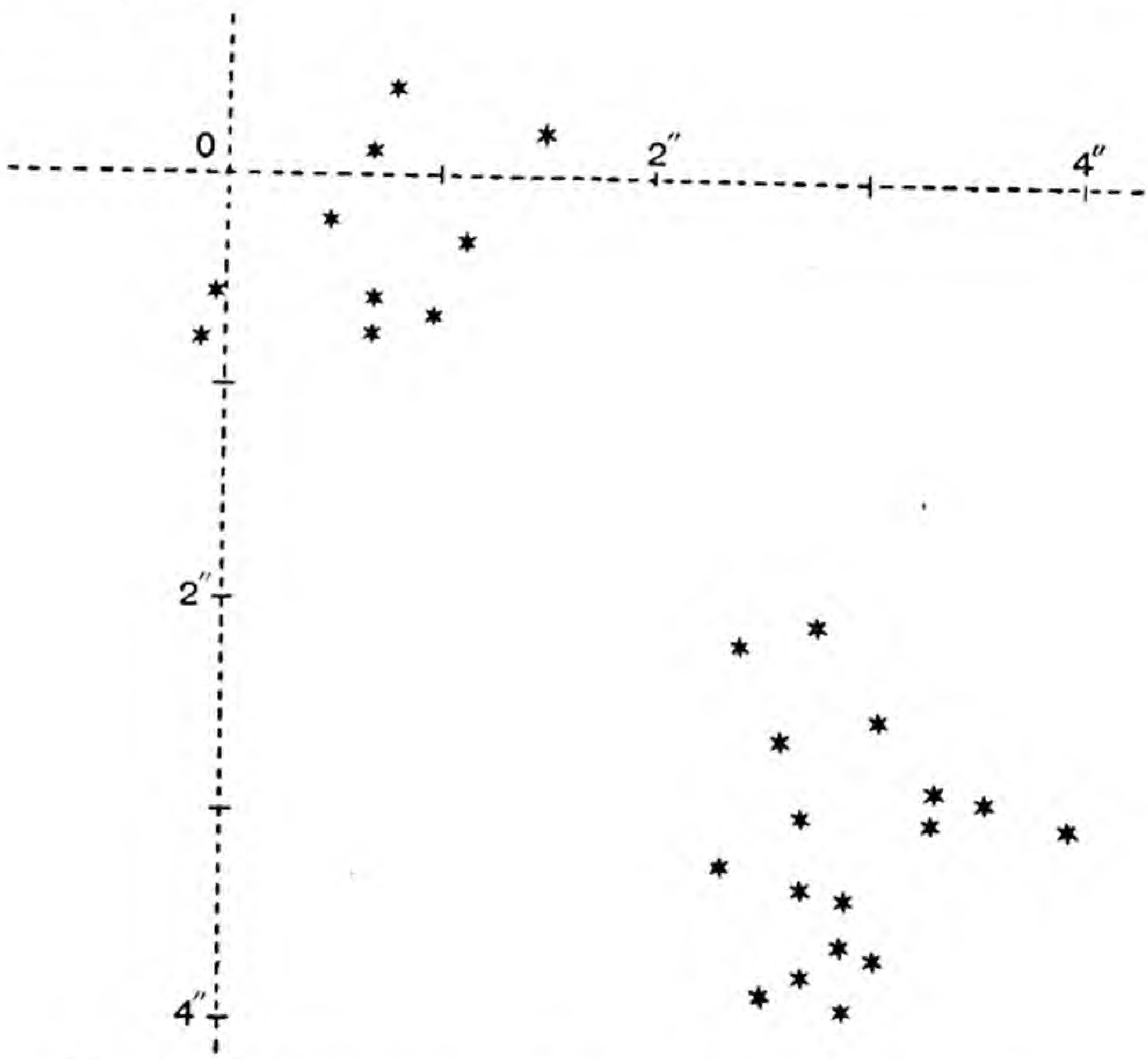


FIG. 4.—Moving Cluster of "Orion" Stars in Perseus.

A group to which the name moving cluster may be applied more legitimately is to be found in the constellation Perseus; it was detected simultaneously by J. C. Kapteyn, B. Boss, and the writer. If we examine all the stars of the Orion type (Type B) in the region of the sky between R.A. 2^h and 6^h and Dec. $+36°$ and $+70°$ (about one-thirtieth of the whole sphere), we shall find that their motions fall into two groups. In Fig. 4 the motion of each star is

denoted by a cross, the star having a proper motion which would carry it from the origin O to the cross in a century. If all the stars were to start from the origin at the same instant with their actual observed proper motions, then after the lapse of a century they would be distributed as shown in the diagram. Only one star has travelled beyond the limits of the figure and is not shown; with this exception the figure includes all the Type B stars in the region for which data are available.

The upper group of crosses, which is close to the origin, consists of stars with very minute proper motions, all less than 1″·5 per century, and scarcely exceeding the probable error of the determinations. These are clearly the very remote stars, and there is not the slightest evidence that they are really associated with one another; they appear to cling together because the great distance renders their diverse motions inappreciable. The lower group consists of seventeen stars sharing very nearly the same motion both as regards direction and magnitude. They evidently form a moving cluster similar in character to those we have considered. Their association is further confirmed by the fact that they are not scattered over the whole area investigated, but occupy a limited region of it.

Table 7 shows the stars which constitute this group. It has been pointed out by T. W. Backhouse [5] that Nos. 742 to 838 form part of a very striking cluster visible to the naked eye. The stars α Persei and σ Persei, which are not of the Orion Type, are included in the visual cluster; the motion of the latter shows that it has no connection with this system, but α Persei appears to belong to it and may therefore be added to the group. The other stars in this part of the sky have also been examined so far as possible, but none of them show any evidence of connection with the moving cluster. All but three of the stars are arranged in a sort of chain, which may indicate a flat cluster (on the plan of the Ursa Major system) seen edge-

ways. It is always likely that some spurious members may be included through an accidental coincidence of motion, and it may be suspected that the three outlying stars are not really associated with the rest; on the other hand, they may well be regarded as original members, which have been more disturbed by extraneous causes than the others.

Owing to the small proper motion, the convergent point of this group cannot well be determined. The motion deviates appreciably from the direction of the solar antapex; so that this cluster possesses some velocity of its own apart from that attributable to the Sun's own motion.

TABLE 7.

Moving Cluster in Perseus.

Boss's No.	Name of Star.	Type.	Mag.	R.A.	Dec.	Centennial Motion.	Direction.
				h. m.	°	"	°
678	Pi. 220	B5	5·6	2 54	+52	4·3	51
740	30 Persei . . .	B5	5·5	3 11	+44	3·8	55
742	29 Persei . . .	B3	5·3	3 12	+50	4·5	52
744	31 Persei . . .	B3	5·2	3 12	+50	4·2	51
767	Pi. 37	B5	5·4	3 16	+49	3·6	45
780	Brad. 476 . . .	B8	5·1	3 21	+49	3·2	47
783	Pi. 56	B5	5·8	3 22	+50	4·9	37
790	34 Persei . . .	B3	4·8	3 22	+49	4·4	57
796	Brad. 480 . . .	B8	6·1	3 24	+48	4·7	54
817	ψ Persei . . .	B5	4·4	3 29	+48	4·3	42
838	δ Persei	B5	3·0	3 36	+47	4·6	51
898	Pi. 186	B5	5·5	3 49	+48	3·9	40
910	ε Persei	B0	2·9	3 51	+40	3·9	49
947	*c* Persei	B3	4·2	4 1	+47	4·4	43
1003	*d* Persei	B3	4·9	4 14	+46	4·5	55
1253	15 Camelopardi .	B3	6·4	5 11	+58	3·5	37
1274	ρ Aurigae . . .	B3	5·3	5 15	+42	4·5	40
772	α Persei	F5	1·7	3 17	+50	3·8	55

In the last column the "direction" is the angle between the direction of motion and the declination circle at 4 hours R.A.

The tracing of these connections between stars widely separated from one another is an important branch of

modern stellar investigation; and, as the proper motions of more stars become determined, it is likely that further interesting discoveries will be made. It seems worth while at this stage to consider what are the exact criteria by which we may determine whether a group of stars possesses that close mutual relation which is denoted by the term "moving cluster." Since some thousands of proper motions are available, it must be possible, if we take almost any star, to select a number of others the motions of which agree with its motion approximately. This is especially the case if, the parallax and radial velocity being unknown, the direction of motion is alone considered; but, even if the velocities were known in all three co-ordinates, we could pick out groups which would agree approximately; just as in a small volume of gas there must be many molecules having approximately identical velocities. Clearly the agreement of the motions is no proof of association, unless there is some further condition which indicates that the coincidence is in some way remarkable. There is a further difficulty that, as we have already seen in Chapter II., stars, scattered through the whole region of the universe that has been studied, show common tendencies of motion, so that they have been divided into the two great star-streams. We must be careful not to mistake an agreement of motion arising from this general cosmical condition for the much more intimate association which is seen in the Taurus and Ursa Major systems.

In the case of the Taurus and Perseus clusters the discrimination is comparatively simple. These are compact groups of stars, so that only a small region of the sky and a small volume of space are considered, and the extraneous stars which might yield chance coincidences are not numerous. In the Taurus-cluster the large amount of the motion makes the group remarkable; and, although in the Perseus-cluster the proper motion is not so great, we have

been careful to show by the diagram that it is very distinctive. In the latter cluster, moreover, the resemblance of its members in type of spectrum helped to render the detection possible.

The Ursa Major system, which is spread over a large part of the sky, presents greater difficulties. The discrimination of its more scattered members was only possible owing to the fact that its motion is in a very unusual direction. Its convergent point is a long way from the apex of either star-stream and from the solar apex, and stars moving in or near that direction are rare. In the writer's investigation of the two star-streams based on Boss's Preliminary General Catalogue, a striking peculiarity was presented in one region, which on enquiry proved to be due to five stars of this system; that five stars moving in this way should attract attention, sufficiently illustrates the fact that motion in this particular direction is exceptional. We are thus to a large extent safeguarded from chance coincidences; nevertheless, our ground is none too certain, and it may reasonably be suspected that one or two of the members at present assigned to the group will prove to be spurious.

When the supposed cluster is not confined to one part of the sky and to one particular distance from the Sun, when there is nothing remarkable in its assigned motion, and when the choice of stars is not sufficiently limited, by the consideration of a particular spectral type or otherwise, not much weight can be attached to an approximate agreement of motion. A careful statistical study of groups in these adverse circumstances may eventually lead to important results; but for the present we cannot be satisfied to admit clusters the credentials of which do not reach the standard that has been laid down.

In concluding this chapter we may try to sum up the importance of the discovery of moving clusters in stellar astromony. An immediate result is that in the Taurus

and Ursa Major stream we have been able to arrive at precise knowledge of the distance, relative distribution, and luminosity of stars which are far too remote for the ordinary methods of measurement to be successful. An important extension of this knowledge may be expected when the proper motions of fainter stars have been accurately determined. Further, the possibility of stars widely sundered in space preserving, through their whole life-time up to now, motions which are equal and parallel to an astonishingly close approximation, is a fact which must be reckoned with when we come to consider the origin and vicissitudes of stellar motions. Generally the stars which show these associations are of early types of spectrum; but in the Taurus cluster there are many members as far advanced in evolution as our Sun, some even of type K, whilst in the more widely diffused Ursa Major system there are three stars of type F. Some of these systems would thus appear to have existed for a time comparable with the life-time of an average star. They are wandering through a part of space in which are scattered stars not belonging to their system—interlopers penetrating right among the cluster stars. Nevertheless, the equality of motion has not been seriously disturbed. It is scarcely possible to avoid the conclusion that the chance attractions of stars passing in the vicinity have no appreciable effect on stellar motions; and that if the motions change in course of time (as it appears they must do) this change is due, not to the passage of individual stars, but to the central attraction of the whole stellar universe, which is sensibly constant over the volume of space occupied by a moving cluster.

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Perseus Cluster	Kapteyn, *Internat. Solar Union*, Vol. 3, p. 215; Benjamin Boss, *Astron. Journ.*, No. 620; Eddington, *Monthly Notices*, Vol. 71, p. 43.

There are in addition two less defined clusters, which have attracted some attention, viz., a large group of Type B stars in Scorpius and Centaurus, and the 61 Cygni group, which consists of stars scattered over most of the sky having a linear motion of the large amount 80 km. per sec.

Scorpius-Centaurus Cluster .	Kapteyn, *loc. cit.*, p. 215; Eddington, *loc. cit.*, p. 39.
61 Cygni Cluster	B. Boss, *Astron. Journ.*, Nos. 629, 633.

CHAPTER V

THE SOLAR MOTION

It was early recognised that the observed motions of the stars were changes of position relative to the Sun, and that part of the observed displacements might be attributed to the Sun itself being in motion. The question "What is the motion of the Sun?" raises at once the philosophical difficulty that all motion is necessarily relative. In reality the manner in which the observed motion is to be divided between the Sun and the star is indeterminate; these bodies are moving in a space absolutely devoid of fixed reference marks, and the choice and definition of a framework of reference that shall be considered at rest is a matter of convention. Probably philosophers of the last century believed that the undisturbed æther provided a standard of rest which might suitably be called absolute; even if at the time it could not be apprehended in practice, it was an ultimate ideal which could be used to give theoretical precision to their statements and arguments. But according to modern views of the æther this is no longer allowable. Even if we do not go so far as to discard the æther-medium altogether, it is generally considered that no meaning can be attached to the idea of measuring motion relative to it; it cannot be used even theoretically as a standard of rest.

In practice the standard of rest has been the "mean of

the stars," a conception which may be difficult to define rigorously, but of which the general meaning is sufficiently obvious. Comparing the stars to a flock of birds, we can distinguish between the general motion of the flock and the motions of particular individuals. The convention is that the flock of stars as a whole is to be considered at rest. It is not necessary now to consider the reasons that may have suggested that the mean of the stars was an absolute standard of rest; it is sufficient to regard it as a conventional standard, which has considerable usefulness. If there is any real unity in the stellar system, we may expect to obtain a simpler and clearer view of the phenomena by referring them to the centroid of the whole rather than to an arbitrary star like the Sun. By the centroid is meant in practice the centre of mass (or rather the centre of mean position) of those stars which occur in the catalogues of proper motions that are being discussed. As it is only the motion of this point that is being considered, its actual situation in space is not of consequence. If the motion of the centroid varied considerably according to the magnitude of the stars used or the particular region of the sky covered by the catalogue, it would be a very inconvenient standard. It is not yet certain what may be the extent of the variations arising from a particular selection of stars; but, as the data of observation have improved, the wide variations shown in the earlier investigations have been much reduced or satisfactorily explained. At the present day, whilst few would assert that the "mean of the stars" is at all a precise standard, the indeterminateness does not seem sufficiently serious to cause much inconvenience.

The determination of the motion of the stars in the mean relative to the Sun, and the determination of the solar motion (relative to the mean of the stars) are two aspects of the same problem. The relative motion, whichever way it is regarded, is shown in our observations by a

strong tendency of the stars to move towards a point in the sky, which according to the best determinations is near β Columbae. Although individual stars may move in widely divergent or even opposite directions, the tendency is so marked that the mean of a very few stars is generally sufficient to exhibit it. Sir William Herschel's[1] first determination in 1783 was made from seven stars only, yet he was able to indicate a direction which was a good first approximation. From his time up till recent years the determination of the solar motion was the principal problem in all statistical investigations of the series of proper motions which were measured from time to time. This investigation was usually associated with a determination of the constant of precession—a fundamental quantity which is closely bound up with the solar motion in the analysis. In fact, both quantities are required to define our framework of reference; the solar motion defines what is to be regarded as a fixed position, and the precession-constant defines fixed directions among the continually shifting stars. The numerous older determinations of the solar motion are now practically superseded by two results published in 1910–11, which rest on the best material yet available.

The determination by Lewis Boss[2] from the proper motions of his *Preliminary General Catalogue of* 6188 *Stars* gives,—

Solar Apex	R.A.	270°·5	±	1°·5
	Dec.	+34°·3	±	1°·3

The determination by W. W. Campbell[3] from the radial velocities (measured spectroscopically) of 1193 stars gives,—

Solar Apex	R.A.	268°·5	±	2°·0
	Dec.	+25°·3	±	1°·8

Speed of the solar motion 19·5 ± 0·6 kilometres per second.

The probable errors are not given by Campbell; but the

foregoing approximate values are easily deduced from the data in his paper.

The discordance in declination between these two results, derived respectively from the transverse and the radial motions, is considerably greater than can be attributed to the accidental errors of the determinations. Possibly the discordance may be attributed to the different classes of stars used in the two investigations. Campbell's result depends almost wholly on stars brighter than $5^{m}\cdot 0$, whereas Boss included all stars to the sixth magnitude and many fainter stars. Moreover, Boss's result depends more especially on the stars nearest to the Sun; for in forming the mean proper motion in any region the near stars (having the largest angular motions) have most effect, whereas in forming the mean radial motion the stars contribute equally irrespective of distance. So far as can be judged, however, these differences will not explain the discordance. Boss made an additional determination of the solar apex, rejecting stars fainter than $6^{m}\cdot 0$; the resulting position R.A. $269^{\circ}\cdot 9$, Dec. $+34^{\circ}\cdot 6$ is almost identical with his main result. The writer,[1] examining the same proper motions on the two star-drift theory, by a method which gives equal weight to the near and distant stars, arrived at the position R.A. $267^{\circ}\cdot 3$, Dec. $+36^{\circ}\cdot 4$, again a scarcely appreciable change. The cause of the difference between the results from the proper motions and the radial motions thus remains obscure.

One of the most satisfactory features of Boss's determination of the solar apex is the accordance shown by the stars of different galactic latitudes. If there is any relative motion between stars in different parts of the sky, it would be expected to appear in a division according to galactic latitude. The following comparison of the results derived from regions of high and low galactic latitudes is given by Boss.

Galactic Latitude of Zones.	Solar Apex. R.A.	Dec.
$-7°$ to $+7°$	269° 40′	+33° 17′
$-19°$ to $-7°$, and $+19°$ to $+7°$	270 55	29 52
$-42°$ to $-19°$, and $+42°$ to $+19°$	269 51	34 18
S. Gal. Pole to $-42°$, and N. Gal. Pole to $+42°$	270 32	36 27

The differences are quite as small as could be expected from the accidental errors.

Another comparison of different areas of the sky can be made from the results of an analysis by the two-drift theory, which has the advantage that the position depends equally on all the stars used, instead of (as in the ordinary method) the nearest stars having a preponderating share. We find,—

Region.		Solar Apex. R.A.	Dec.
Polar Area	Dec. $+36°$ to $+90°$ / Dec. $-36°$ to $-90°$	265°·5	+37°·0
Equatorial Area	Dec. $-36°$ to $+36°$	269°·4	+36°·4

There is thus a very satisfactory stability in the position of the apex determined from different parts of the sky.* The evidence is less certain as to its dependence on the magnitude and spectral type of the stars. There is some indication that the declination of the apex tends to increase for the fainter stars; but it is not entirely conclusive. The range of magnitude in Boss's catalogue is scarcely great enough to provide much information; so far as it goes, it is opposed to the view that there is any alteration in the apex for stars of different magnitudes, for, as already mentioned, the stars brighter than $6^{m}\cdot0$ give a result almost identical with that derived from the whole catalogue. From the Groombridge stars (Dec. $+38°$ to N. Pole), F. W. Dyson and W. G. Thackeray [5] found—

* In the foregoing comparisons antipodal regions have always been taken together. There remains a possibility of a discordance between opposite hemispheres.

Magnitude.	Solar Apex.		No. of Stars.
m. m.	R.A.	Dec.	
1·0–4·9	245°	+16°·0	200
5·0–5·9	268°	+27°·0	454
6·0–6·9	278°	+33°·0	1,003
7·0–7·9	280°	+38°·5	1,239
8·0–8·9	272°	+43°·0	811

This shows a steady increase in declination with diminishing magnitude. It must, however, be noted that the area covered by the Groombridge Catalogue is particularly unfavourable for a determination of the declination of the apex.

Other evidence pointing in the same direction has been found by G. C. Comstock,[6] who made a determination of the solar motion from 149 stars of the ninth to twelfth magnitudes. He was able to obtain proper motions of these stars, because they had been measured micrometrically as the fainter companions of double stars, but had been found to have no physical connection with the principal stars. The resulting position of the apex is

R.A. 300° Dec. +54°

In a more recent investigation [7] the same writer has used 479 faint stars, with the results

Magnitude	$7^m·0$ to $10^m·0$	Apex	R.A. 280° Dec. +58°
,,	$10^m·0$,, $13^m·0$	,,	R.A. 288° Dec. +71°

The weight of these determinations cannot be great, but they tend to confirm the increase of declination with the faintness of the stars.

Earlier investigations in which the stars were classified by magnitude are those of Stumpe and Newcomb. The former, using stars of large proper motion only, found a considerable progression in declination with faintness. Newcomb, on the other hand, who used stars of small proper motion only, found that the declination is steady. We now know that, owing to the phenomenon of star-streaming, the exclusion of stars above or below certain limits of motion is not legitimate, so that the contradictory character of these two results is not surprising.

The comparative uncertainty of the proper motions of the fainter stars requires that results based on them should be received with caution. In particular, since the mean distance of the stars increases with faintness, the average parallactic motion becomes smaller, and a systematic error in the declinations of any zone has a greater effect on its apparent direction. This is particularly serious, because these investigations have been usually based on northern stars only or on an even less extensive region.

There is fairly consistent evidence that the declination of the solar apex depends to some extent on the spectral type of the stars, being more northerly for the later types. In Boss's investigation [8] the following results were found,—

Type.	Solar Apex. R.A.	Solar Apex. Dec.	No. of Stars.
Oe5—B5	274°·4	+34°·9	490
B8—A4	270°·0	28°·3	1,647
A5—F9	265°·9	28°·7	656
G	259°·3	42°·3	444
K	275°·4	40°·3	1,227
M	273°·6	38°·8	222

The later types G, K, M thus yield a declination differing markedly from the earlier types; or, if we prefer to set aside the results for the groups containing few stars, which may be subject to large accidental errors, and confine attention to types A (B8 – A4) and K, the difference of 12° between the results of these two classes is evidently significant.

The results of Dyson and Thackeray from the Groombridge stars show the same kind of progression.

Type.	Solar Apex. R.A.	Solar Apex. Dec.	No. of Stars.
B, A	269°	+23°	1100
F, G, K	273°	37°	866

Other investigations of this relation depend mainly on stars now included in Boss's catalogue, and used in his

discussion. It therefore does not seem necessary to quote them.

To sum up the results we have arrived at, it appears that we can assign a point in the sky at about R.A. 270°, Dec. +34° towards which the motion of the Sun relative to the stellar system is directed. For some reason at present unknown, the determinations of this point by means of the spectroscopic radial velocities differ appreciably from those based on the transverse motions, giving a declination nearly 10° lower than the point mentioned. When different parts of the sky are examined the results are generally in good agreement, so that there can be little relative motion of the stars as a whole in different regions. There is some evidence that the solar apex increases in declination as successively fainter stars are considered, and it seems certain that for the later types of spectrum the declination is higher than for the earlier types. From all causes the solar apex from a special group of stars may (apart from accidental error) range from about +25° to +40° in declination; variations in right ascension appear to be small and accidental.

The speed with which the Sun moves in the direction thus found can only be measured from the radial motions. The result derived from the greatest amount of data is 19·5 km. per sec.

Attention has been lavished on the investigation of the solar motion, not only on account of its intrinsic interest, but also because it is a unit of much importance in many investigations of the distribution of the more distant stars. The annual or centennial motion of the Sun is a natural unit of comparison in dealing with the stellar system, generally superseding the radius of the earth's orbit, which is too small to be employed except for a few of the nearest stars. It provides a far longer base-line than can be obtained in parallax-observations; for the annual motion of the Sun amounts to four times the radius of the earth's

orbit, and the motion of fifty or a hundred years, or even longer, may be used. The apparent displacement of the star attributable to the solar motion is called the *parallactic motion.* By determining the parallactic motion (in arc) of any class of stars their average distance can be found, just as the distance of an individual star is found from its annual parallax. It is not possible to find by observation the parallactic motion of an individual star, because it is combined with the star's individual motion; but for a group of stars which has no systematic motion relative to the other stars, these individual motions will cancel in the mean.

It may be appropriate to add some remarks on the theory of the determination of the solar motion from the observations. The method usually adopted for discussing a series of proper motions is that known as Airy's.

Take rectangular axes, Ox being directed to the vernal equinox, Oy to R.A. 90°, and Oz to the north pole. The parallactic motion (opposite to the solar motion) may be represented by a vector, with components X, Y, Z, directed to the solar antapex. X, Y, Z are supposed to be expressed in arc, so as to give the parallactic motion of a star at a distance corresponding to the mean parallax of all the stars considered.

Taking a small area of the sky let the mean proper motion of the stars in the area be μ_α, μ_δ in right ascension and declination respectively. Then considering the projection of (X, Y, Z) on the area considered, we have

$$-X \sin \alpha + Y \cos \alpha = \mu_\alpha$$
$$-X \cos \alpha \sin \delta - Y \sin \alpha \sin \delta + Z \cos \delta = \mu_\delta$$

where it is assumed that these stars are at the same mean distance as the rest, and that their individual motions cancel out. If these assumptions are not exactly satisfied, the deviations are likely to be mainly of an accidental

character. Taking the above equations for each region, a least-squares solution may then be made to determine X, Y, Z. The right ascension and declination A, D of the solar antapex are given by

$$\tan A = Y/X$$
$$\tan D = Z/(X^2+Y^2)^{\frac{1}{2}}$$

Additional terms in the equations of condition involving the correction to the precessional constant and to the motion of the equinox are often inserted, but these need not concern us here. When stars distributed uniformly over the whole sky are considered, the additional terms have no effect on the result.

Although the argument is clearer when we use the mean proper motion over an area for forming equations of condition, it is quite legitimate to use each star separately. For it is easily seen that the resulting normal equations are practically identical in the two procedures. In using the mean proper motion, it is easier and more natural to give equal weights to equal areas of the sky instead of weighting according to the number of stars; this is generally an advantage. Further the numerical work is shortened.

There are two weak points in Airy's method. First the mean proper motion (which, if not formed separately for each area, is virtually formed in the least-squares solution) is generally made up of a few large motions and a great number of extremely small ones. It is therefore a very fluctuating quantity, the presence or omission of one or two of the largest motions making a big difference in the mean. In a determination based nominally on 6000 stars, the majority may play only a passive part in the result, and the accuracy of the result is scarcely proportionate to the great amount of material used. The second point is more serious, since it leads to systematic error. We have assumed that the mean parallax of the stars in each area differs only by accidental fluctuations from the mean of

the whole sky; but this is not the case. The stars near the galactic plane have a systematically smaller parallax than those near the galactic poles.

It has often been recognised that this property of the galactic plane may cause a systematic error in the apex derived from the discussion of a limited part of the sky. Perhaps it is not so generally known that it will also cause error even when the whole sky is used. It seems worth while to examine this point at length. Happily it turns out that the error is not very large, but this could scarcely have been foreseen.

If the mean parallax in any area is p times the average parallax for the whole sky, we may take account of the variation with galactic latitude by setting

$$p = 1 + \epsilon P_2(\cos\theta)$$

where θ is the distance from the galactic pole, ϵ is a coefficient, and

$$P_2(\mu) = \tfrac{1}{2}(3\mu^2 - 1).$$

We shall consider the case when the observations extend uniformly over the whole sky.

The equations of condition should then read

$$-Xp\sin a + Yp\cos a = \mu_a$$
$$-Xp\cos a\sin\delta - Yp\sin a\sin\delta + Zp\cos\delta = \mu_\delta.$$

We wish to re-interpret the results of an investigator who has not taken p into account. We therefore form normal equations, just as he would do, viz., from the right ascensions:

$$X\Sigma p\sin^2 a - Y\Sigma p\sin a\cos a = -\Sigma\mu_a\sin a$$
$$-X\Sigma p\sin a\cos a + Y\Sigma p\cos^2 a = \Sigma\mu_a\cos a.$$

and from the declinations:

$$X\Sigma p\cos^2 a\sin^2\delta + Y\Sigma p\sin a\cos a\sin^2\delta - Z\Sigma p\cos a\sin\delta\cos\delta = -\Sigma\mu_\delta\cos a\sin\delta$$
$$X\Sigma p\sin a\cos a\sin^2\delta + Y\Sigma p\sin^2 a\sin^2\delta - Z\Sigma p\sin a\sin\delta\cos\delta = -\Sigma\mu_\delta\sin a\sin\delta$$
$$-X\Sigma p\cos a\sin\delta\cos\delta - Y\Sigma p\sin a\sin\delta\cos\delta + Z\Sigma p\cos^2\delta = \Sigma\mu_\delta\cos\delta$$

giving the combined equations:

$$X\Sigma p(\sin^2 a+\cos^2 a\sin^2\delta)-Y\Sigma p\sin a\cos a\cos^2\delta-Z\Sigma p\cos a\sin\delta\cos\delta=-\Sigma(\mu_a\sin a+\mu_\delta\cos a\sin\delta)$$

$$-X\Sigma p\sin a\cos a\cos^2\delta+Y\Sigma p(\cos^2 a+\sin^2 a\cos^2\delta)-Z\Sigma p\sin a\sin\delta\cos\delta=\Sigma(\mu_a\cos a-\mu_\delta\sin a\sin\delta)$$

$$-X\Sigma p\cos a\sin\delta\cos\delta-Y\Sigma p\sin a\sin\delta\cos\delta+Z\Sigma p\cos^2\delta=\Sigma\mu_\delta\cos\delta.$$

Now it clearly can make no difference in a least-squares solution whether we resolve our proper motions in right ascension and declination or in galactic latitude and longitude. The value of the solar motion, which makes the sum of the squares of the residuals in R.A. and Dec. a minimum, must be the same as that which makes the sum of the squares of the residuals in Gal. Lat. and Long. a minimum. We may therefore treat a solution as though it had been made in galactic co-ordinates, although the actual work was done in equatorial co-ordinates.

Let then a, δ now stand for galactic longitude and latitude, so that X, Y, Z is the parallactic motion vector referred to rectangular galactic co-ordinates. We shall have

$$p=1+\tfrac{1}{2}\epsilon(3\sin^2\delta-1).$$

Taken over a whole sphere the mean value of

$$\begin{aligned}p(\sin^2 a+\cos^2 a\sin^2\delta) &= \frac{2}{3}+\frac{1}{15}\epsilon\\ p(\cos^2 a+\sin^2 a\sin^2\delta) &= \frac{2}{3}+\frac{1}{15}\epsilon\\ p\cos^2\delta &= \frac{2}{3}-\frac{2}{15}\epsilon.\end{aligned}$$

The other coefficients vanish when integrated over a sphere. Thus the normal equations become (setting N for the total number of stars used)

$$\begin{aligned}\frac{2}{3}X\left(1+\frac{1}{10}\epsilon\right) &= -\Sigma(\mu_a\sin a+\mu_\delta\cos a\sin\delta)\div N\\ \frac{2}{3}Y\left(1+\frac{1}{10}\epsilon\right) &= \Sigma(\mu_a\cos a-\mu_\delta\sin a\sin\delta)\div N\\ \frac{2}{3}Z\left(1-\frac{1}{5}\epsilon\right) &= \Sigma(\mu_\delta\cos\delta)\div N.\end{aligned}$$

And if X_0, Y_0, Z_0 are the solutions obtained when the P_2 term is neglected,

$$X\left(1+\frac{1}{10}\epsilon\right)=X_0 \quad Y\left(1+\frac{1}{10}\epsilon\right)=Y_0 \quad Z\left(1-\frac{1}{5}\epsilon\right)=Z_0.$$

The original and corrected galactic latitudes of the antapex being λ_0, λ, we have

$$\tan\lambda=\frac{\left(1+\frac{1}{10}\epsilon\right)}{\left(1-\frac{1}{5}\epsilon\right)}\tan\lambda_0,$$

whilst the galactic longitude is unaltered.

The effect of the decrease of parallax towards the galactic plane is thus to make λ_0 numerically less than λ. The uncorrected position of the solar apex is too near the galactic plane.

Inserting numerical values, $\lambda_0=20°$, and ϵ may perhaps be $\frac{1}{3}$ (*i.e.*, mean parallax at the pole / mean parallax in the plane $=8/5$), we find $\lambda=21°57'$. The correction is just under 2°. Reverting to equatorial co-ordinates, the correction is mainly in right ascension, the right ascension given by the ordinary solution being about 2°·4 too great.

It is quite practicable to work out the corresponding corrections, when the proper motions cover only a zone of the sky limited by declination circles. In this case we have to retain equatorial co-ordinates throughout, and express $P_2(\cos\theta)$ in terms of α and δ. The mean values of the functions of $\sin\alpha$, $\cos\alpha$, $\sin\delta$ and $\cos\delta$ that occur are readily evaluated for the portion of the sphere used. As the numerical work depends on the particular zone chosen, we shall not pursue this matter further.

A second method of finding the solar apex from the proper motions, known as Bessel's method, has been used by H. Kobold.[9] Each star is observed to be moving along a great circle on the celestial sphere. Consider the poles of these great circles. If the stellar motions all converged to a point on the sphere, the poles would all lie along the

great circle equatorial to that point. Thus a tendency of the stars to move towards the solar antapex should be indicated by a crowding of the poles towards the great circle equatorial to the antapex. This affords a means of finding the direction of the solar motion by determining the plane of greatest concentration of the poles. It is to be noted, however, that this method makes no discrimination between the two ways in which a star may move along its great circle. Two stars moving in exactly opposite directions will have the same pole. A paradoxer might argue that, as the effect of the solar motion is to cause a minimum number of stars to move towards the solar *apex*, the solar motion will be indicated by a tendency of the poles to avoid the plane equatorial to its direction. However, if the individual motions are distributed according to the law of errors, the crowding to the plane will be found to outweigh the avoidance, so that the method is legitimate though perhaps a little insensitive. But if the individual motions follow some other law, the result may be altogether incorrect. In the light of modern knowledge of the presence of two star-streams, Bessel's method can no longer be regarded as an admissible way of finding the solar apex; but it is interesting historically, for in Kobold's hands it first foreshadowed the existence of the peculiar distribution of stellar motions which is the subject of the next chapter.

The determination of the solar motion from the radial velocities presents no difficulty. If (X, Y, Z) is the vector representing the parallactic motion in linear measure, each star yields an equation of condition:

$$X \cos \alpha \cos \delta + Y \sin \alpha \cos \delta + Z \sin \delta = \text{radial velocity}.$$

A least-squares solution is then made, the individual motions of the stars being treated as though they were accidental errors. The numerical work can be shortened by using in the equations of condition the mean radial velocity for a

small area of the sky, instead of the individual results. The resulting normal equations are practically unaltered, and there is no theoretical advantage.

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CHAPTER VI

THE TWO STAR STREAMS

THE observed motion of any star can be regarded as compounded of two parts; one part, which is attributable to the motion of the Sun as point of reference, is the parallactic motion; and the other residual part is the star's *motus peculiaris* or individual motion. It must be borne in mind that this division cannot generally be effected in practice for the proper motion of a star; because, although the parallactic motion in linear measure is known, we cannot tell how much it will amount to in angular measure, unless we know the star's distance, and this is very rarely the case. On the other hand, the spectroscopic radial velocities, being in linear measure, can always be freed from the parallactic motion, if desired. As the greater part of our knowledge of stellar movements is derived from the proper motions, we cannot study the *motûs peculiares* directly, but must deduce the phenomena respecting them from a statistical study of the whole motions.

In researches on the solar motion, it has usually, though not always, been assumed that the *motûs peculiares* of the stars are at random. This was the natural hypothesis to make, when nothing was known as to the distribution of these residual motions; and certainly, when we consider

how vast are the spaces which isolate one star from its neighbour, and how feeble must be any gravitational forces exerted across such distances, it might well seem improbable that any general tendency or relation could connect the individual motion of one star with another. Yet many years ago the phenomenon of local drifts of stars, or, as they are now called, Moving Clusters, was known. But although such instances of departure from the strict law of random distribution of motions must have been recognised as occurring exceptionally, probably few astronomers doubted that the hypothesis was substantially correct. In 1904, however, Prof. J. C. Kapteyn[1] showed that there is a fundamental peculiarity in the stellar motions, and that they are not even approximately haphazard. This deviation is not confined to certain localities, but prevails throughout the heavens, wherever statistics of motions are available to test it.

Instead of moving indiscriminately in all directions, as a random distribution implies, the stars tend to move in two favoured directions. It does not matter whether the parallactic motion is eliminated or not. A tendency to move in *one* favoured direction would disappear, when the parallactic motion was removed; but a tendency in *two* directions can only be an intrinsic property of the individual motions of the stars. It may seem strange that this striking phenomenon was so long overlooked by those who were working on proper motions; but usually investigators, having the solar motion mainly in their minds, as a first step towards gathering their data into a manageable form grouped together the stars in small regions of the sky, and used the mean motion. This unfortunately tends to conceal any peculiarity in the individual motions. In order to exhibit the phenomenon it is necessary to find some means of showing the statistics of the separate stellar motions; this may be done conveniently in the following way.

ANALYSIS OF PROPER MOTIONS

Confining our attention to a limited area of the sky, so that the apparent motions are seen projected on what is practically a plane, we count up the number of stars observed to be moving in the different directions. If in classifying directions we proceed by steps of 10°, we shall then form a table of the number of stars moving in 36 directions, towards position angles 0°, 10°, 20° . . . 350°.

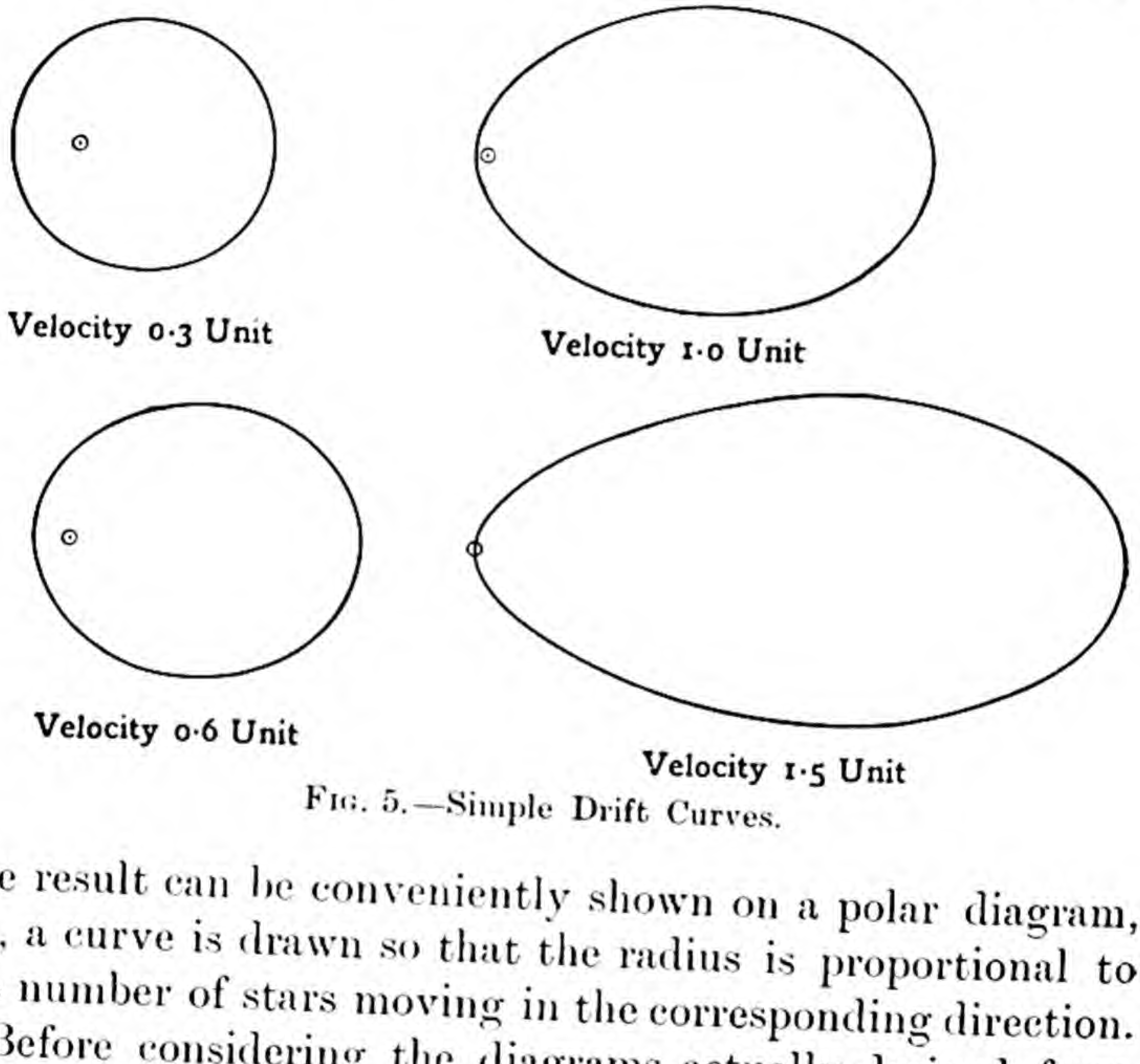


FIG. 5.—Simple Drift Curves.

The result can be conveniently shown on a polar diagram, *i.e.*, a curve is drawn so that the radius is proportional to the number of stars moving in the corresponding direction.

Before considering the diagrams actually derived from observation, let us examine what form of curve would be obtained if the hypothesis of random motions were correct. The curve would not be a circle because of the parallactic motion; for, as these observed motions are referred to the Sun, there would be superposed on the random individual motions the motion of the star-swarm as a whole. If, for example, this latter motion were towards the north, then

clearly there would be a maximum number of stars moving north and fewest south, the number falling off symmetrically on either side from north to south. The exact form can be calculated on the hypothesis of random distribution; it varies with the magnitude of the parallactic motion compared with the average *motus peculiaris*, being more elongated the greater the parallactic motion. In Fig. 5

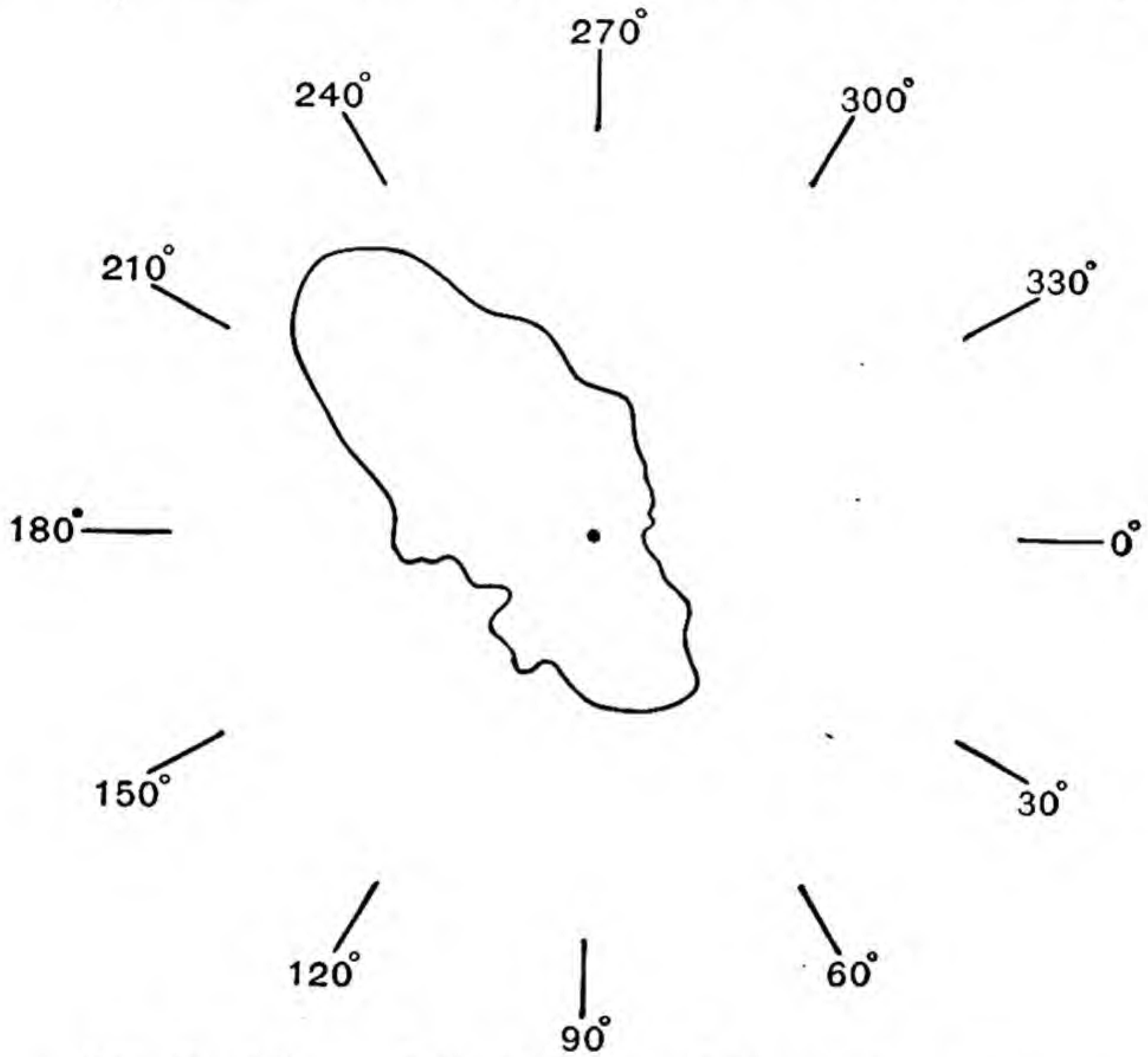


FIG. 6.—Observed Distribution of Proper Motions. (Groombridge Catalogue—R.A. 14h to 18h, Dec. +38° to +70°.)

examples of this curve are given; it may be noted how sensitive is the form of the curve to a small change in the parallactic velocity. It is convenient to have a name for a system such as is represented in these figures, in which the individual motions are haphazard, but the system as a whole is in motion relative to the Sun; we call such a system a *drift*.

As an example of a curve representing the observed distribution of proper motions we take Fig. 6.[2] This

corresponds to a region of the sky between R.A. 14^h and 18^h, Dec. $+38°$ and $+70°$, the proper motions being taken from Dyson and Thackeray's "New Reduction of Groombridge's Catalogue." The motions of 425 stars are here summarised. It is quite clear that none of the single drift curves of Fig. 5 can be made to fit this curve derived from observation. Its form (having regard to the position of the origin) is altogether different. No one of the theoretical curves corresponds to it in even the roughest manner. It will be noticed that there are two favoured directions of motion; the stars are streaming in directions 80° and 225°, the latter being the more pronounced elongation. The actual number of stars moving in each direction is given in the fifth column of Table 8 below. Neither of the favoured directions coincides with that towards the solar antapex, viz., 205°, for this part of the sky. It is true the mean motion of all these stars is towards the antapex, but we see that that is merely a mathematical average between the two partially opposed streams that are revealed in the diagram.

It is possible to obtain a theoretical figure that will correspond approximately with Fig. 6 in the following way. Suppose, instead of the single drift we have hitherto considered, there are two star-drifts. Let one, consisting of 202 stars, be moving in the direction 225° with velocity* 1·20, and the other, consisting of 232 stars, be moving in the direction 80° with the much smaller velocity 0·45. The corresponding curves are P and Q, Fig. 7. If these were seen mingled together in the sky, the resulting distribution would be represented by the curve R. Each radius of R is, of course, formed by adding together the corresponding radii of P and Q. If R is carefully compared with the observed curve, it will be seen that the resemblance is close. The numerical

* The unit velocity $1/h$, which is related to the mean individual motion, is defined in the mathematical theory in the next chapter.

comparisons, which these diagrams illustrate, are given in Table 8; it is there shown that by adding together two

TABLE 8.

Analysis of Proper Motions in the Region R.A. 14ʰ to 18ʰ, Dec. +38° to +70°.

Direction.	Calculated.			Observed.	Difference. Obs. – Calc.
	Drift I.	Drift II.	Total.		
5°	0	6	6	4	–2
15	0	7	7	5	–2
25	0	8	8	6	–2
35	0	10	10	9	–1
45	0	11	11	10	–1
55	0	12	12	14	+2
65	0	12	12	14	+2
75	0	13	13	14	+1
85	0	13	13	13	0
95	0	12	12	12	0
105	1	12	13	10	–3
115	1	11	12	11	–1
125	1	10	11	10	–1
135	1	8	9	10	+1
145	2	7	9	7	–2
155	3	6	9	9	0
165	5	6	11	9	–2
175	7	5	12	14	+2
185	11	4	15	14	–1
195	15	4	19	16	–3
205	19	3	22	21	–1
215	23	3	26	27	+1
225	24	3	27	29	+2
235	23	3	26	26	0
245	19	3	22	19	–3
255	15	3	18	17	–1
265	11	3	14	12	–2
275	7	3	10	11	+1
285	5	3	8	11	+3
295	3	3	6	8	+2
305	2	3	5	7	+2
315	1	3	4	6	+2
325	1	4	5	6	+1
335	1	4	5	5	0
345	1	5	6	5	–1
355	0	6	6	4	–2
Totals . .	**202**	**232**	**434**	**425**	—

theoretical drifts, the observed distribution of motions is approximately obtained. Without pressing the conclusion that a combination of two simple star-drifts will represent the actual distribution in all its detail, we may at least assert that it represents its main features, whereas not even the roughest approximation can be obtained with

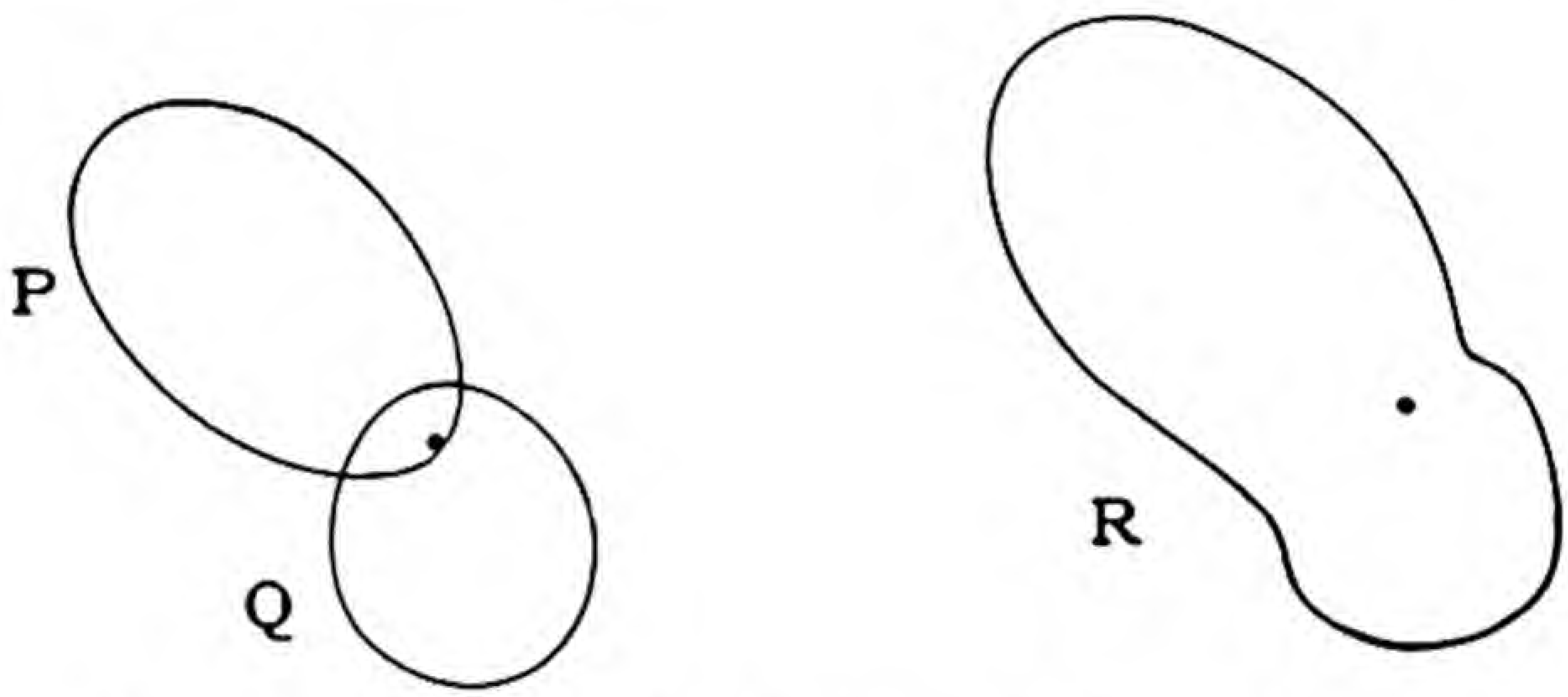


FIG. 7.—Calculated Distributions of Proper Motions.

a single drift, *i.e.*, with the hypothesis of random motions.

As another illustration, we may take Fig. 8, which refers to a different part of the sky, the proper motions this time being taken from Boss's Preliminary General Catalogue. The uppermost curve, which has so interesting an appearance, is derived from the observed proper motions. Curve *B* is the best approximation that can be found on the assumption of a random distribution of motions *plus* the parallactic motion. It may be remarked that since the solar apex is a rather well determined point, the direction of elongation of the curve *B* is not arbitrary; it is necessary to draw it pointing in the known direction of the parallactic motion. The curve *C* is an approximation by a combination of two star-drifts; these again were not taken as arbitrary in direction, but were made to point towards appropriate apices deduced from a general discussion of the whole sky. It is quite probable that there are differences between *A* and *C* which are not purely

accidental, but it will at least be admitted that, whereas the curve *B* bears scarcely any resemblance to the observed curve, the curve *C* reproduces all the main features of the distribution, and from it we can if we choose proceed to investigate the irregularities of detail.

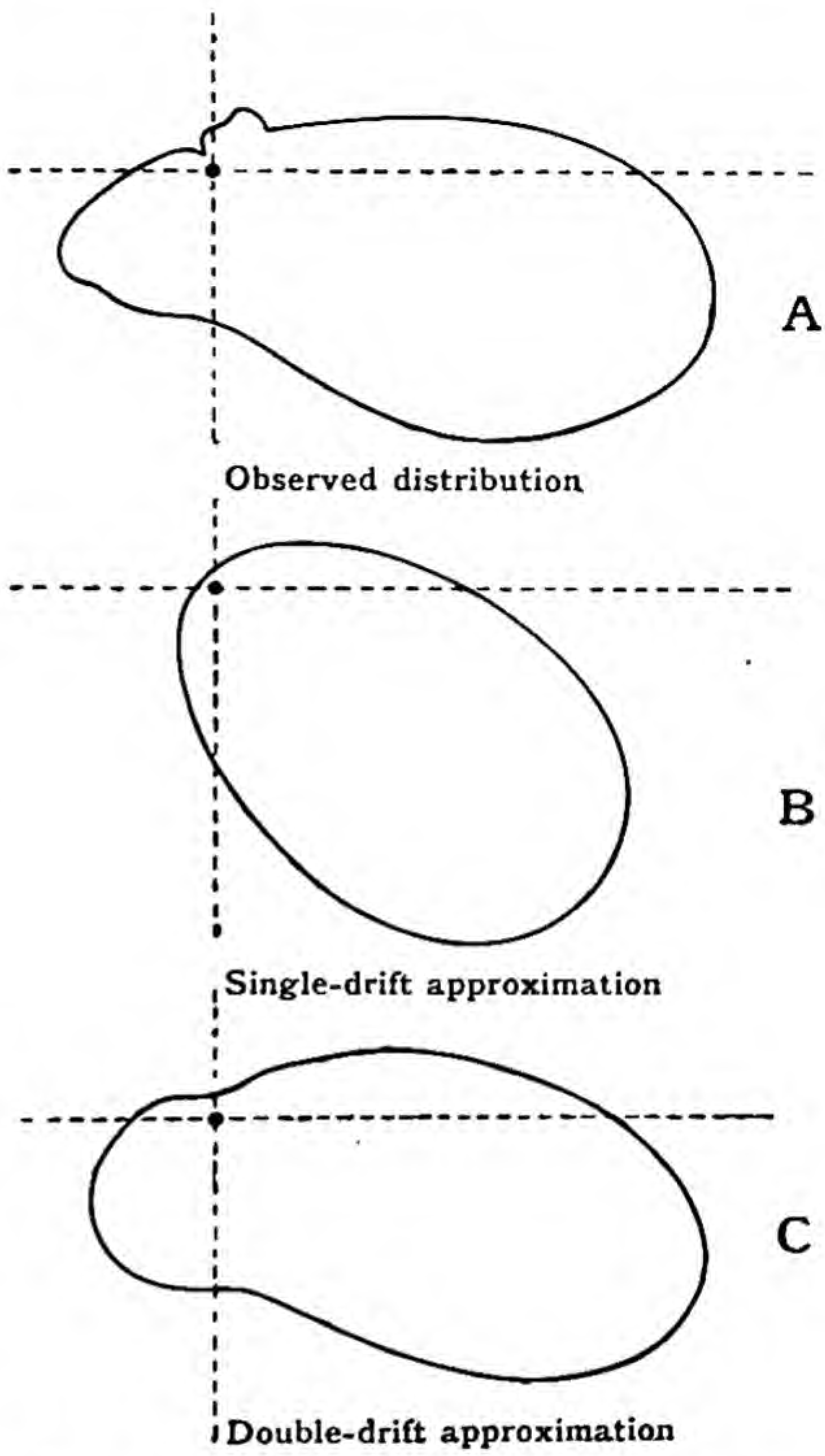


FIG. 8.—Observed and Calculated Distributions of Proper Motions. (Boss, Region VIII.)

The foregoing examples illustrate a method of analysis which has been applied successfully to a great many parts of the sky. It consists in finding, generally by trial and error, a combination of two drifts which will give a distribution of motions agreeing closely with that actually observed. In comparing the results obtained from different parts of the sky, it must be remembered that we are studying the two-dimensional projections of a three-dimensional phenomenon, and the diagrams will vary in appearance as the circumstances of projection vary. The most accurate series of proper motions at present available is contained in Lewis Boss's "Preliminary General Catalogue," and it is of special interest to examine fully the results derivable from it.[3] The catalogue contains 6188 stars well-distributed over the whole sky; practically

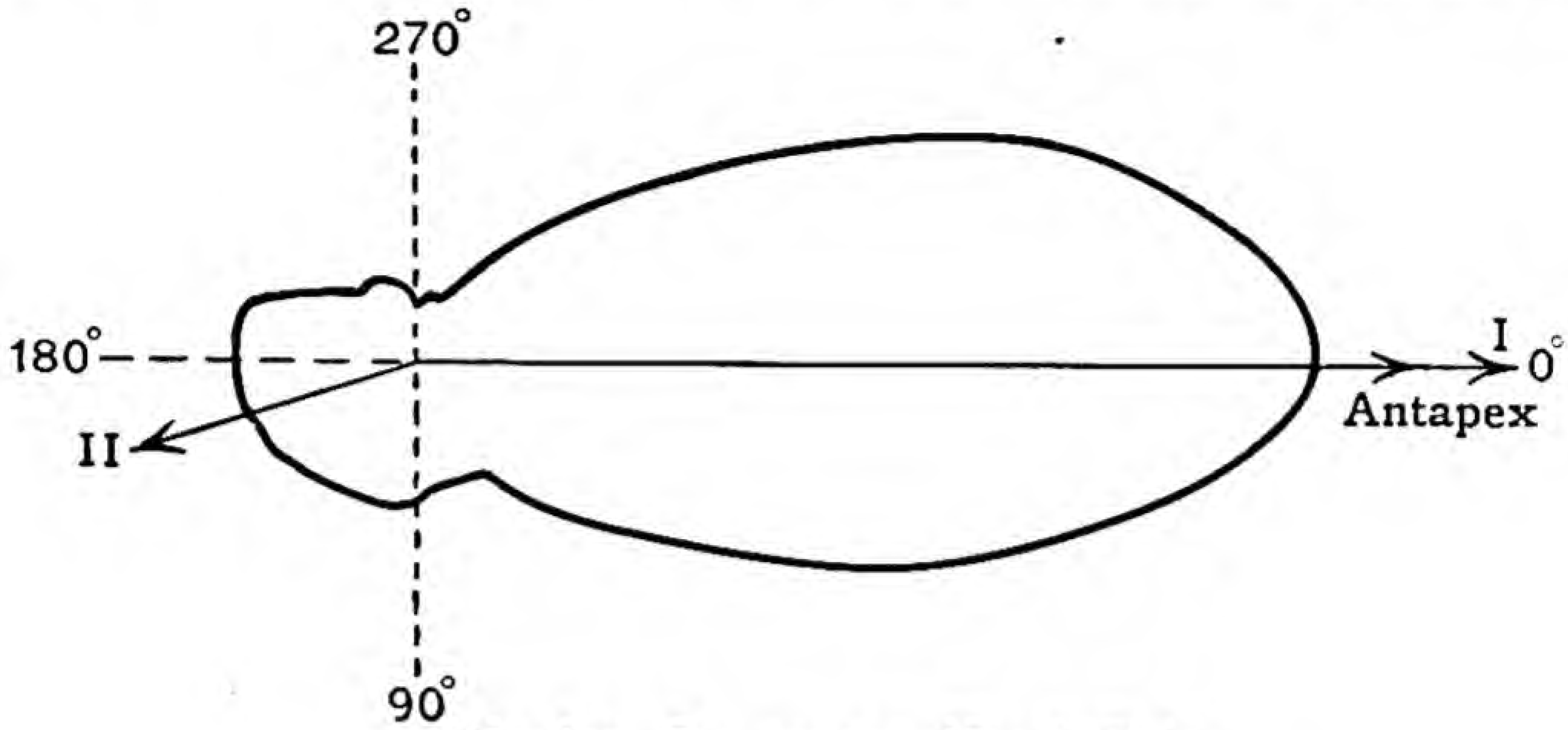
all stars down to the sixth magnitude are included, and the fainter stars appear to be fairly representative and have not been selected on account of the size of their proper motions. A very high standard has been attained in the elimination of systematic error—the main cause of trouble in these researches—though no doubt there is still a possibility of improvement in this respect. There can be no doubt that the catalogue represents the best data that it is at present possible to use.

After excluding certain classes* of stars for various reasons, 5322 remained for investigation. These were divided between seventeen regions of the sky, each region consisting of a compact patch in the northern hemisphere together with an antipodal patch in the southern hemisphere. By taking opposite areas together in this way we double the number of stars without unduly extending the region, for the circumstances of projection in opposite regions are identical. The regions were numbered from I. to XVII., I. being the circular area Dec. +70° to the Pole; II. to VII. formed the belt between +36° and +70° with centres at 0^h, 4^h, 8^h, 12^h, 16^h, and 20^h respectively; and VIII. to XVII. formed the belt 0° to +36° with centres at $1^h\ 12^m$, $3^h\ 36^m$, etc. (These are the positions of the northern portions; the antipodal part of the sky is also to be included in each case.)

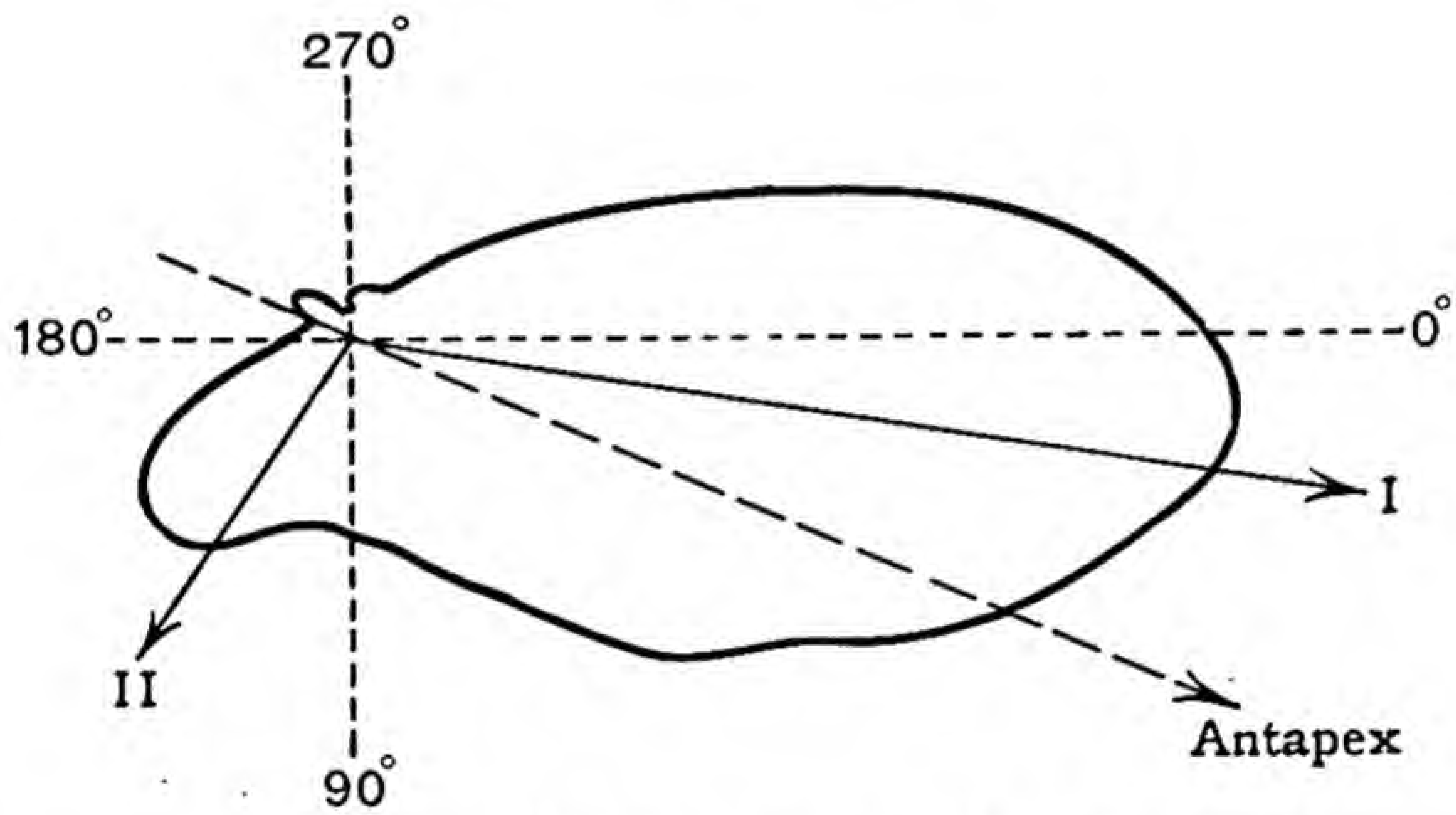
The diagrams for 11 of the 17 regions are given in Fig. 9. The arrows marked Antapex point to the antapex of the solar motion (R.A. 90°·5, Dec. −34°·3); the arrows I. and II. point to the apices of the two drifts, found from the collected results of this discussion. It will be seen that the evidence for the existence of two star-streams is very strong. The tendency to move in the directions of the arrows I. and II. is plainly visible, and it is scarcely

* Viz., stars of the Orion type, members of moving clusters, and the fainter components of binary systems.

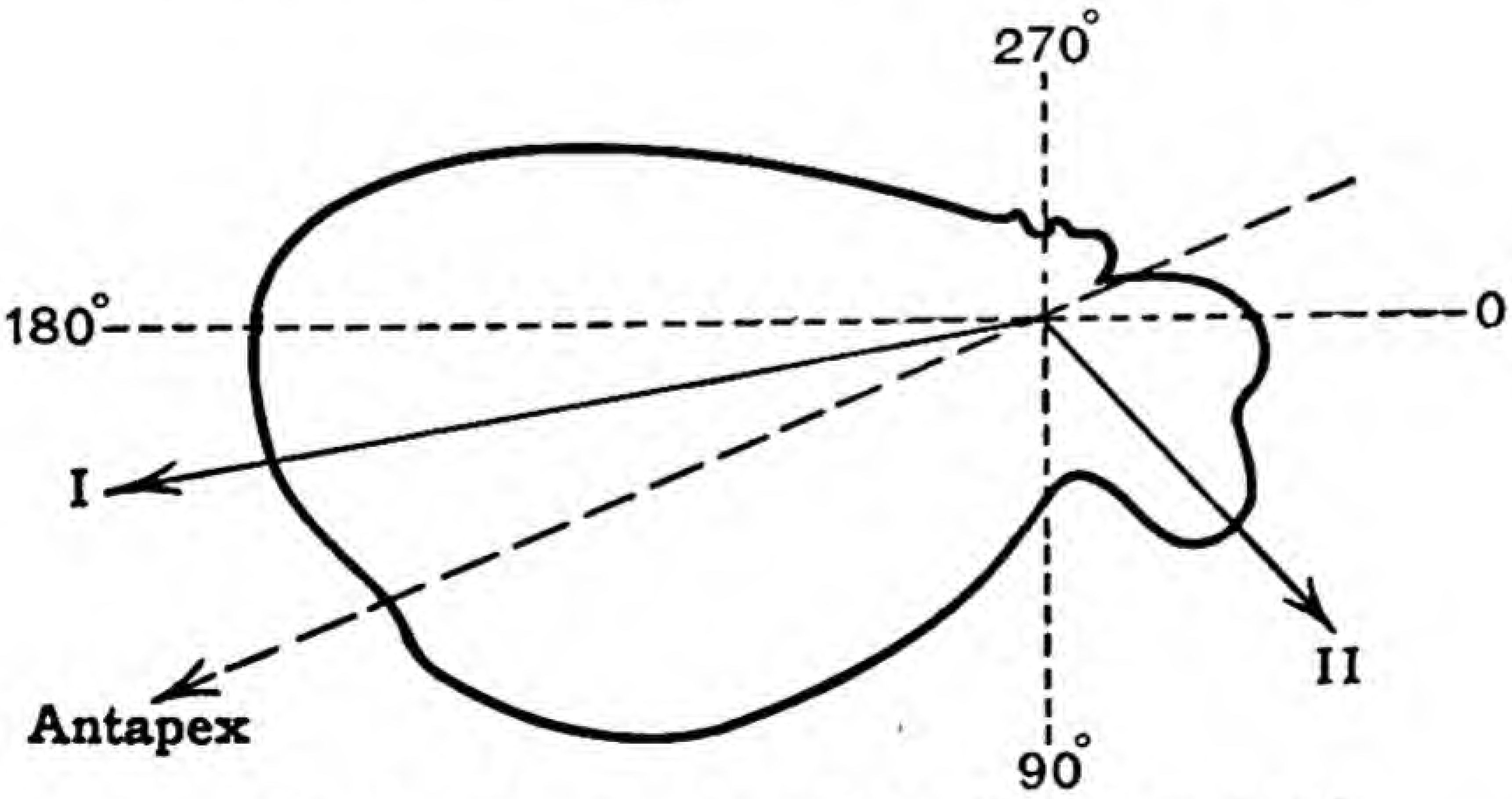
necessary again to emphasise that there is no resemblance to a symmetrical single-drift curve pointing along the antapex arrow. In certain cases, notably Regions XIV.



I. Region I.; centre, North Pole.

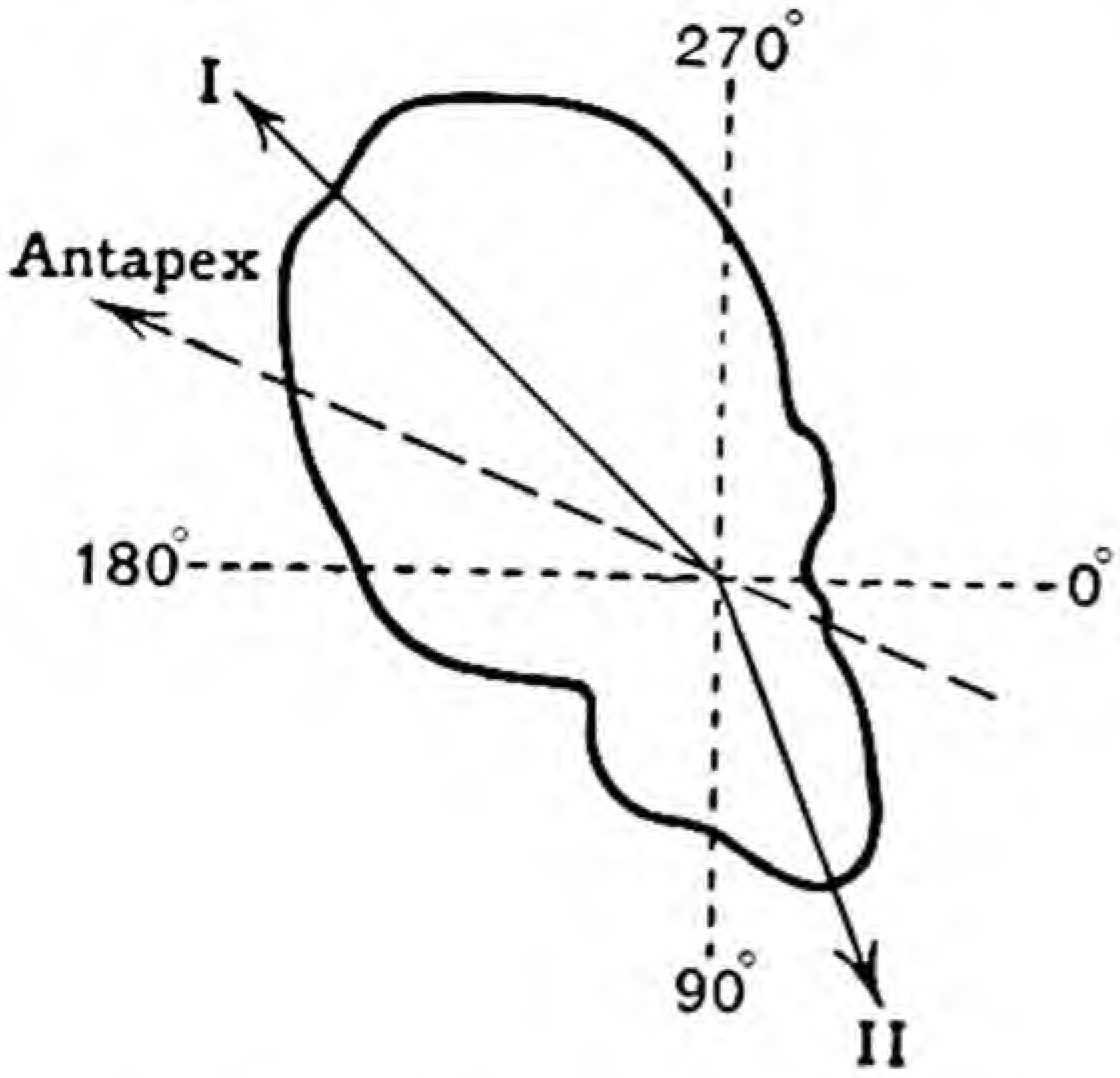


II. Region II.; centre, R.A. 0^h, Dec. +50°.

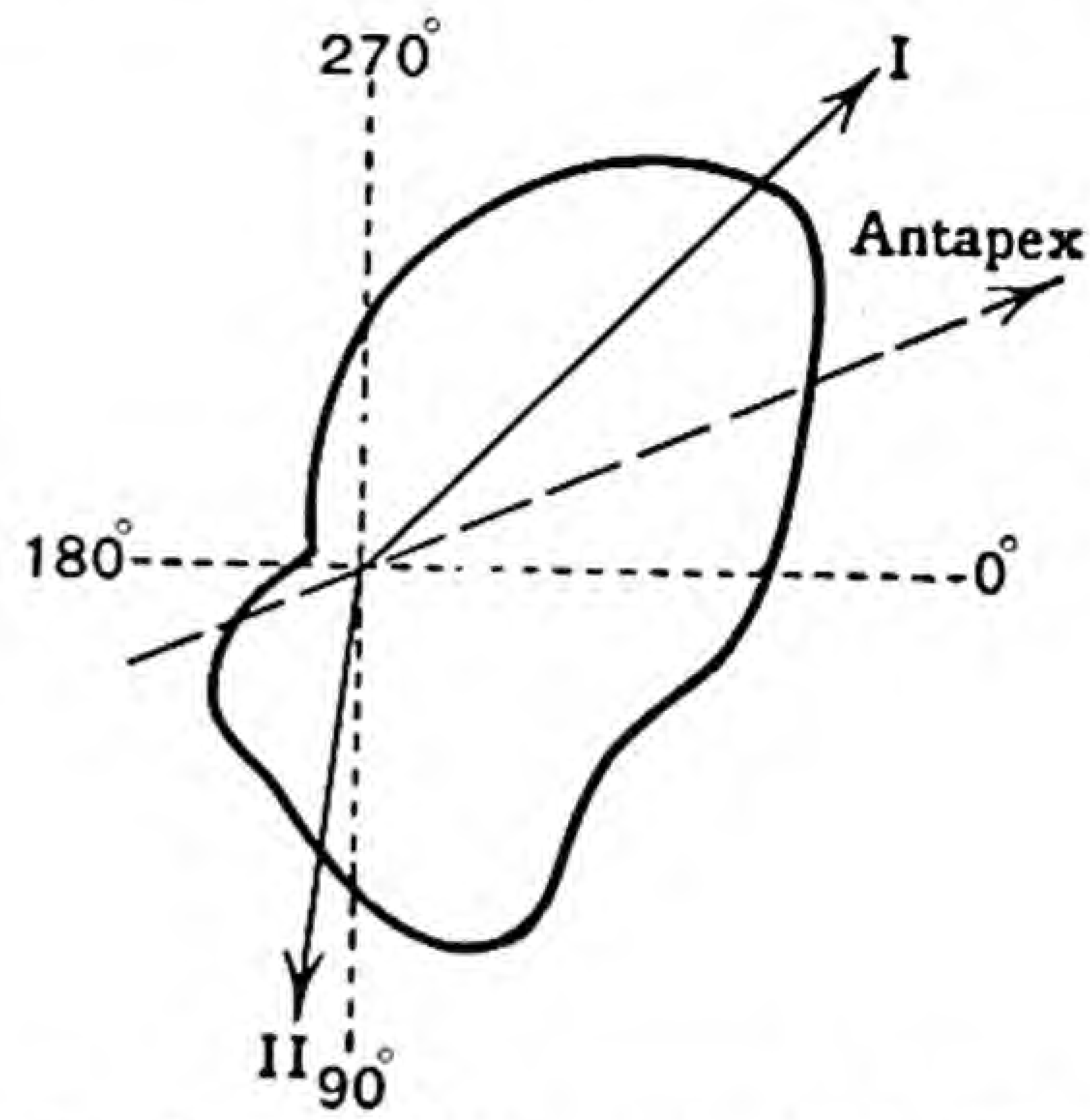


III. Region V.; centre, R.A. 12^h, Dec. +50°.

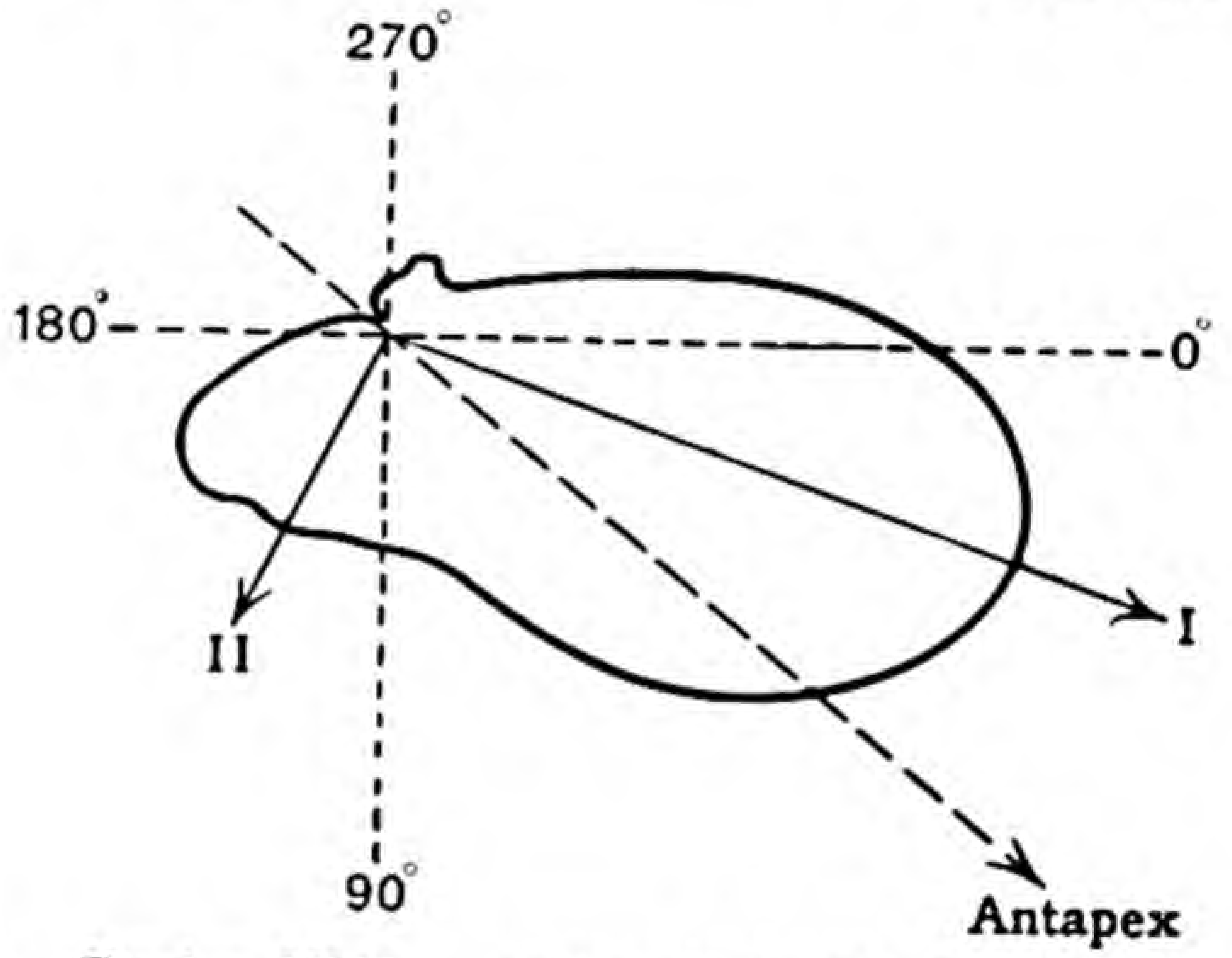
Fig. 9.—Diagrams for the Proper Motions of Boss's "Preliminary General Catalogue."



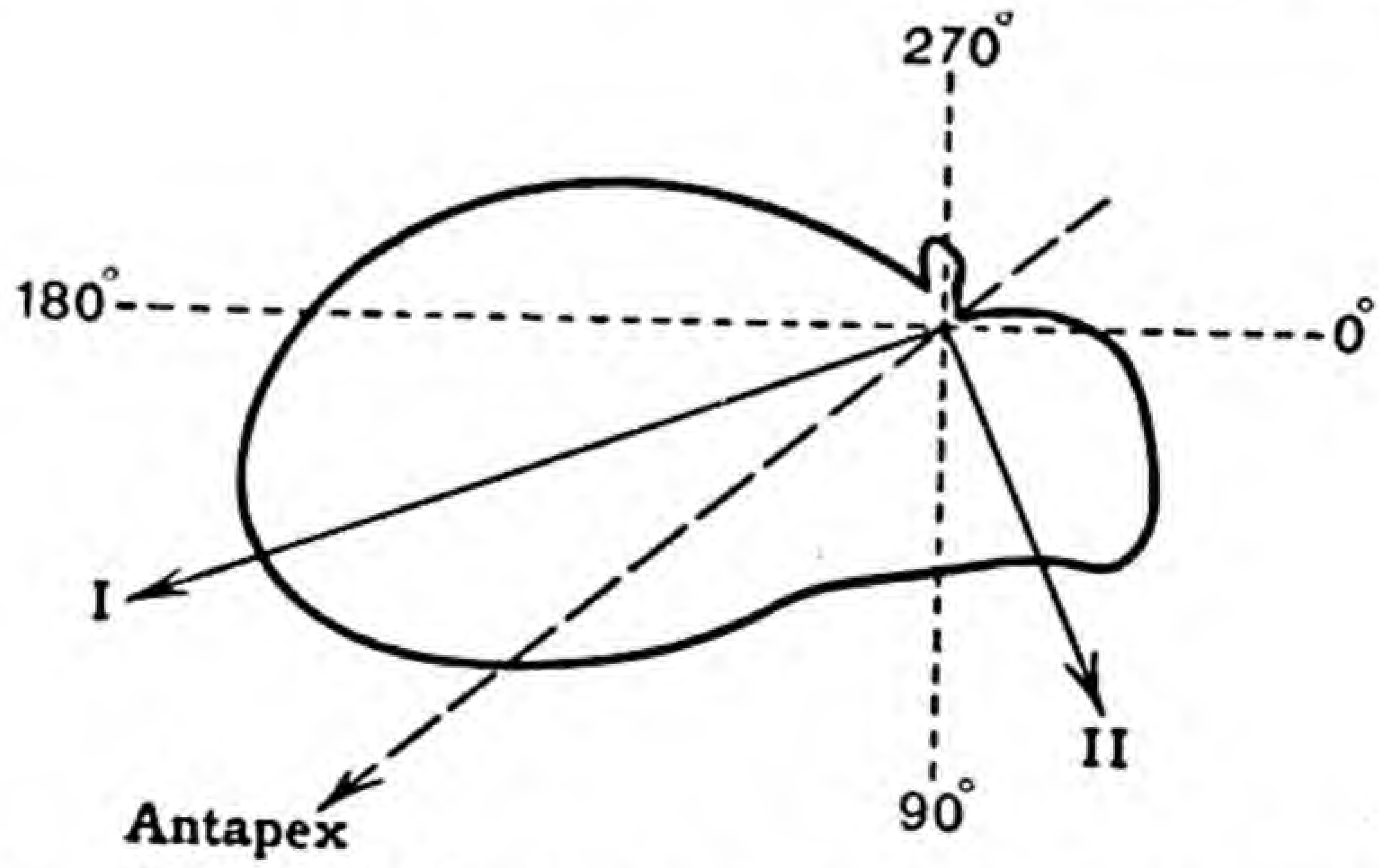
IV. Region VI. ; centre R.A. 16^h, Dec. $+50°$.



V. Region VII. ; centre R.A. 20^h, Dec. $+50°$.

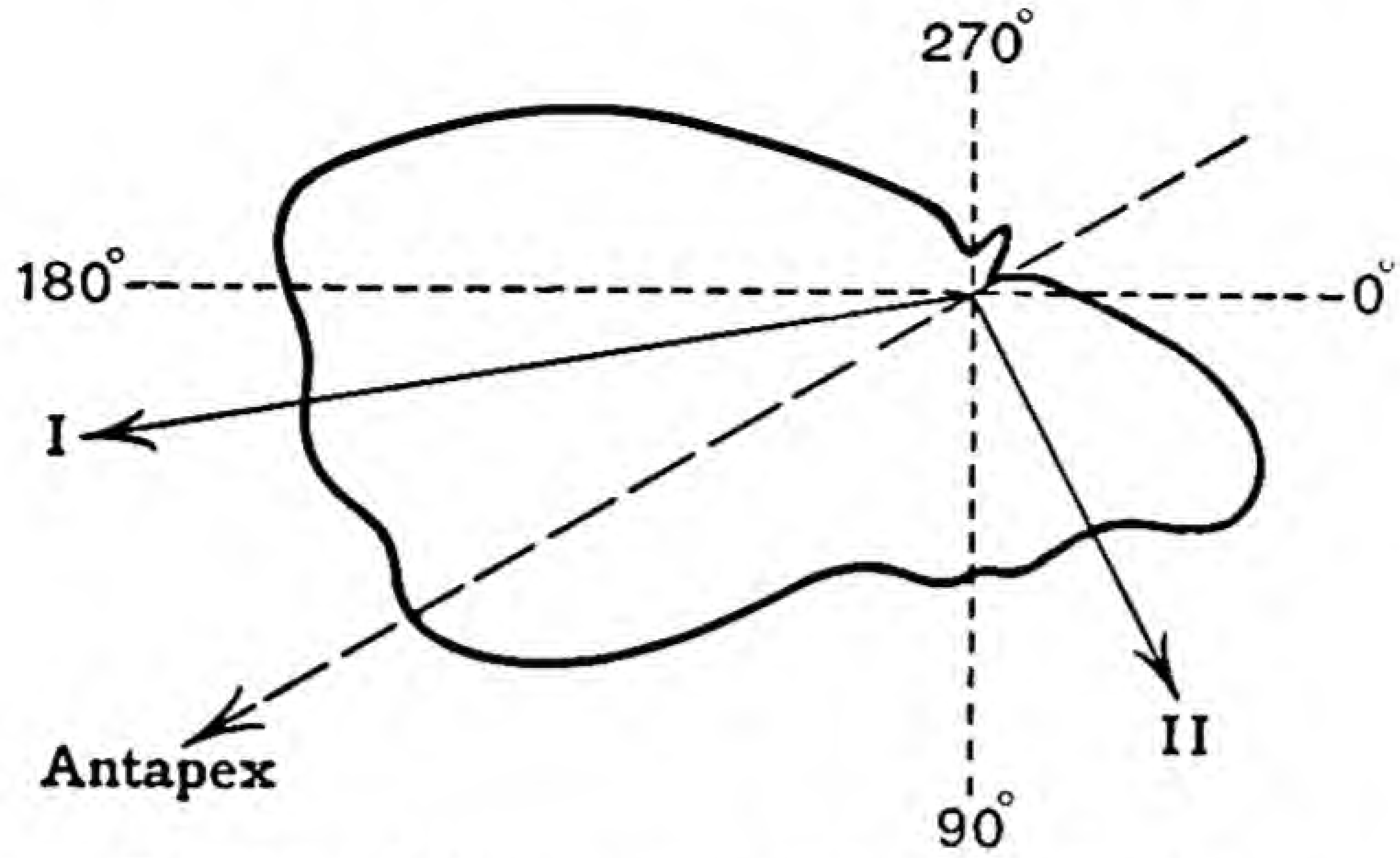


VI. Region VIII. ; centre, R.A. 1^h 12^m, Dec. $+17°$.

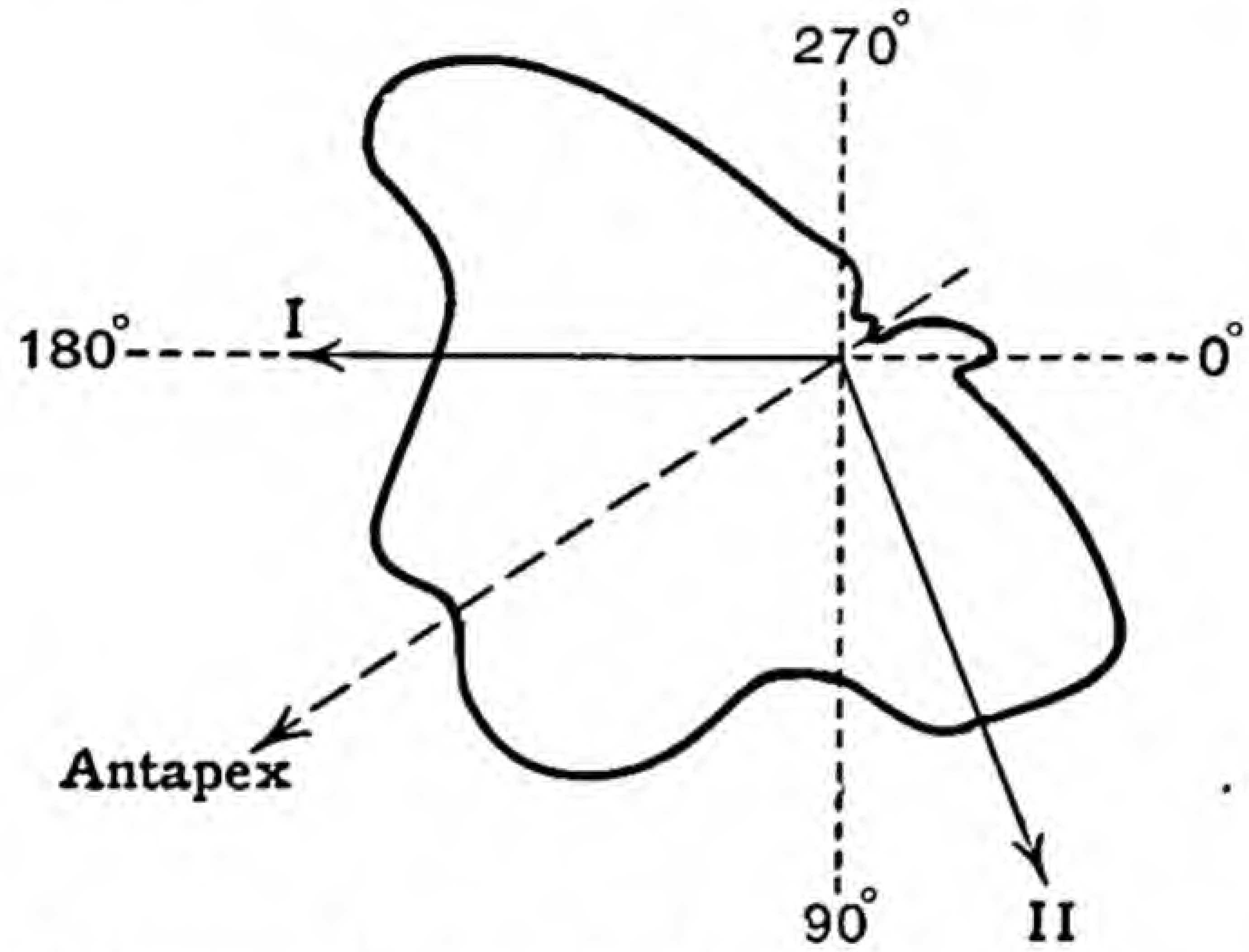


VII. Region XII. ; centre, R.A. 10^h 48^m, Dec. $+17°$.

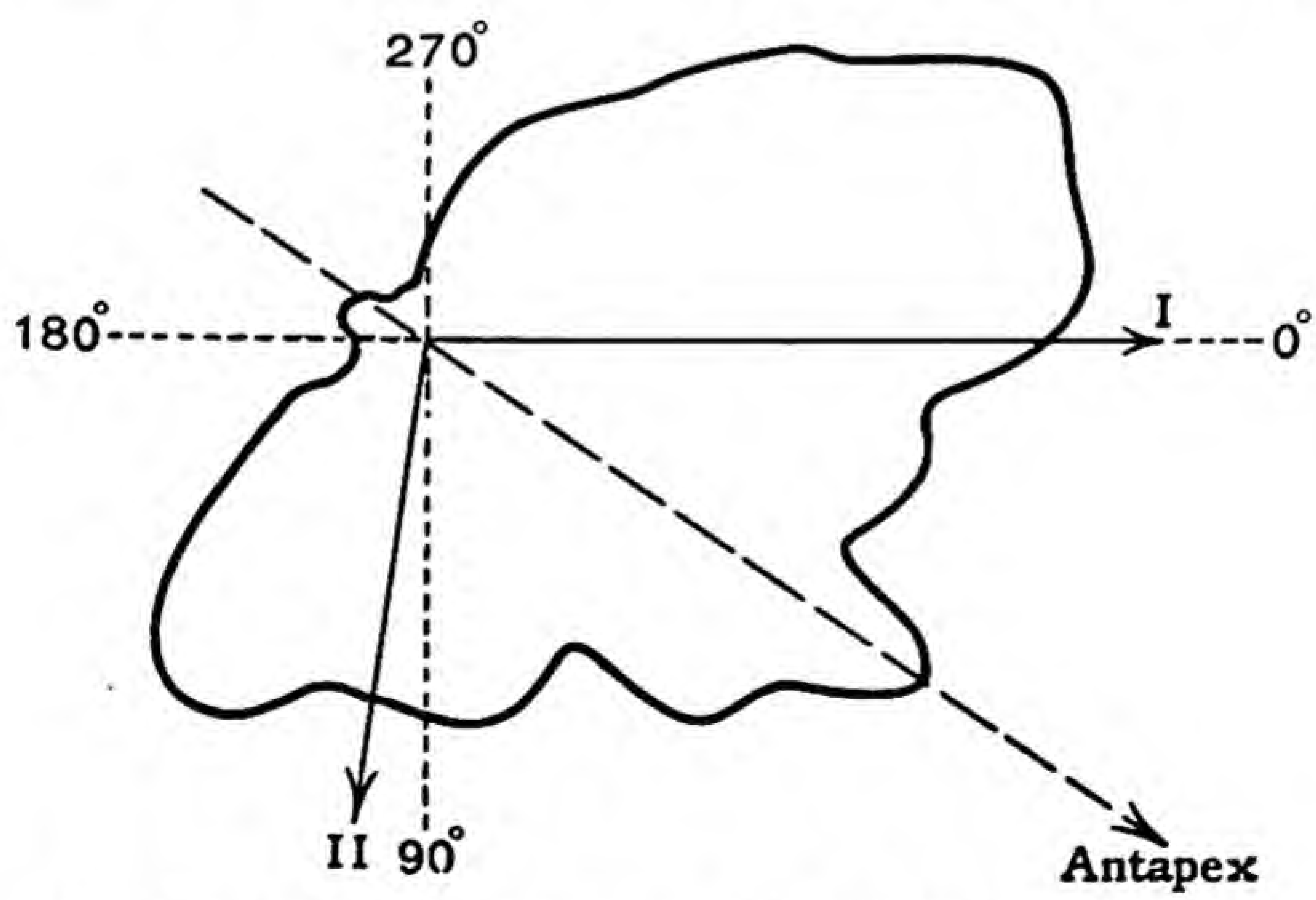
FIG. 9 (*continued*).—Diagrams for the Proper Motions of Boss's "Preliminary General Catalogue."



VIII. Region XIII. ; centre R.A. 13h 12m, Dec. +17°.

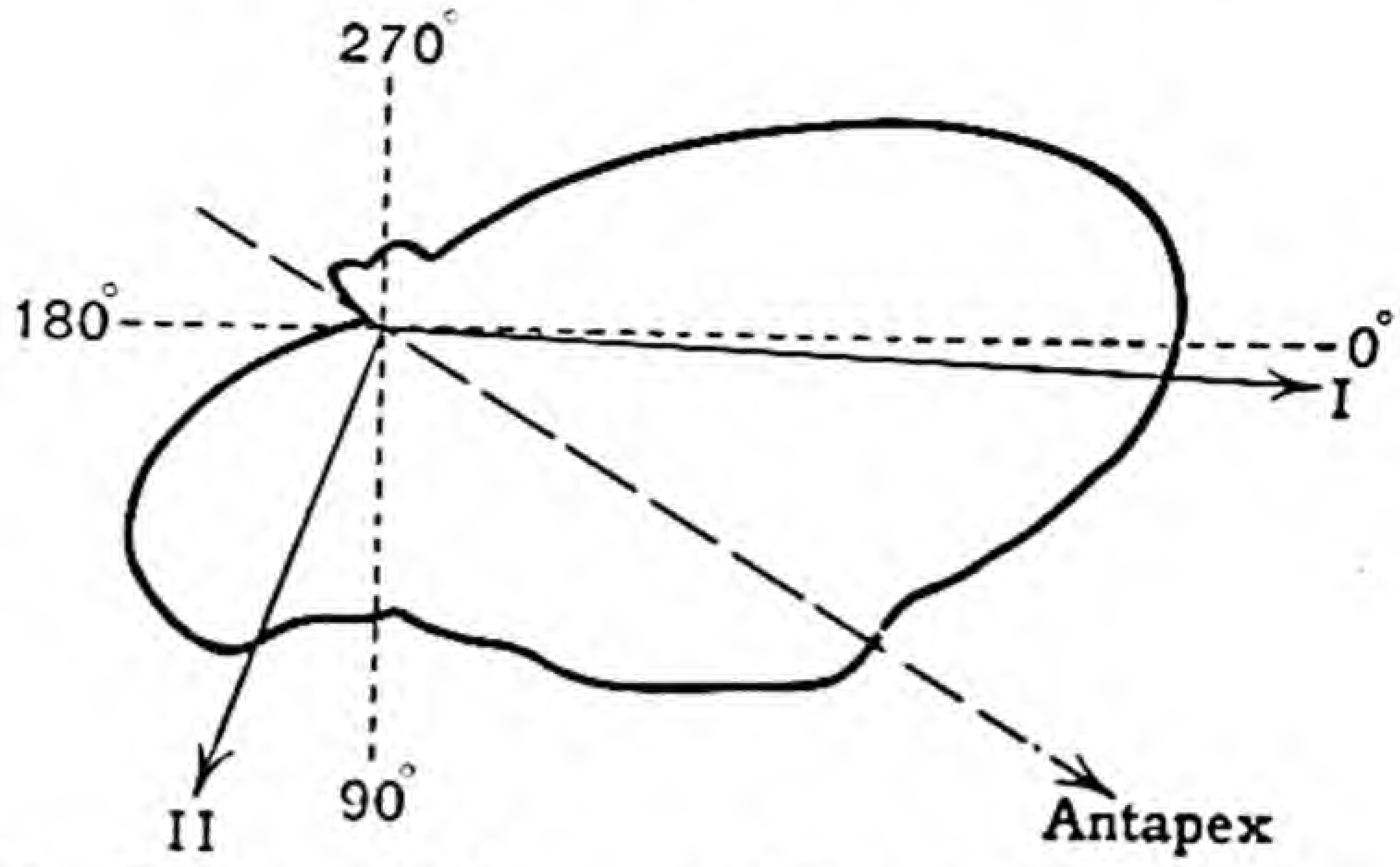


IX. Region XIV. ; centre R.A. 15h 36m, Dec. +17°.



X. Region XVI. ; centre R.A. 20h 24m, Dec. +17°.

Fig. 9 (*continued*).—**Diagrams for the Proper Motions of Boss's "Preliminary General Catalogue."**



XI. Region XVII.; centre R.A. 22^h 48^m, Dec. +17°.

FIG. 9 (*continued*).—Diagrams for the Proper Motions of Boss's "Preliminary General Catalogue."

and XVI., there appears to be a streaming towards the antapex *in addition to* the streaming in the directions of the two drifts, so that the curve appears three-lobed—like a clover leaf. This is an important qualification of our conclusion, but for the present we shall not discuss it; later it will be considered fully. The eleven regions chosen for representation are those in which the separation into two drifts ought to be most plainly indicated. It will be understood that there are parts of the sky in which the projection is such that they are not very plainly separated. In fact there must be one plane of projection on which the drifts have identical transverse motions, and would therefore become indistinguishable, except by having recourse to the radial velocities. The fact, then, that in the regions which are not here represented the phenomenon is shown less plainly, in no way weakens the argument, but rather confirms it.

Let us suppose now that we have succeeded in analysing the stellar motions in each of the seventeen regions into their constituent drifts, and have thus determined the directions and velocities of the two drifts at seventeen points of the celestial sphere. If the drift motion in each region is really the same motion seen in varying projections, we must find that on plotting the directions on a

globe they will all converge to one point. This will be true for each drift separately. The convergence actually found is shown in Figs. 10 and 11. Imagine the great circles traced

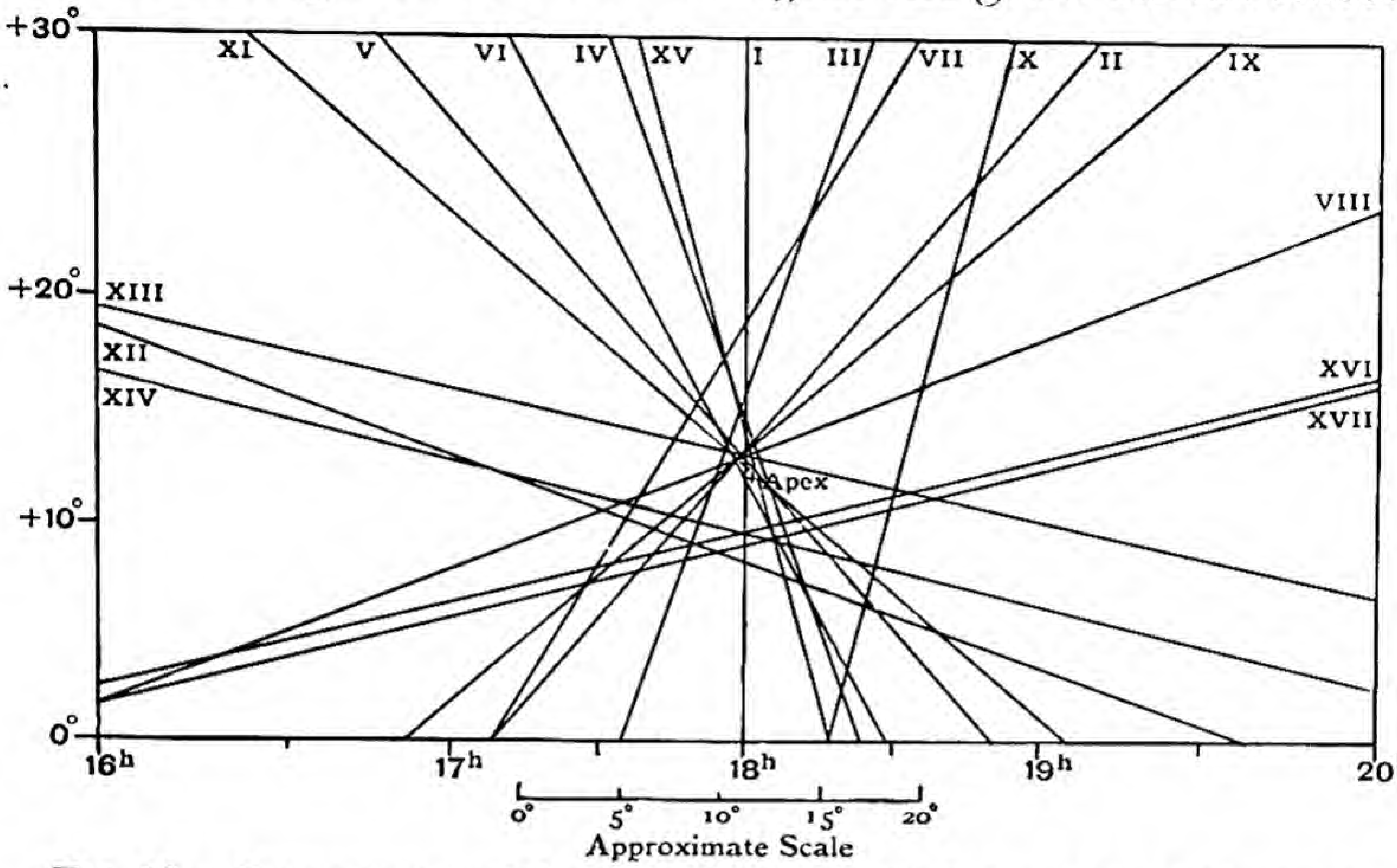


FIG. 10.—Convergence of the Directions of Drift I. from the 17 Regions.

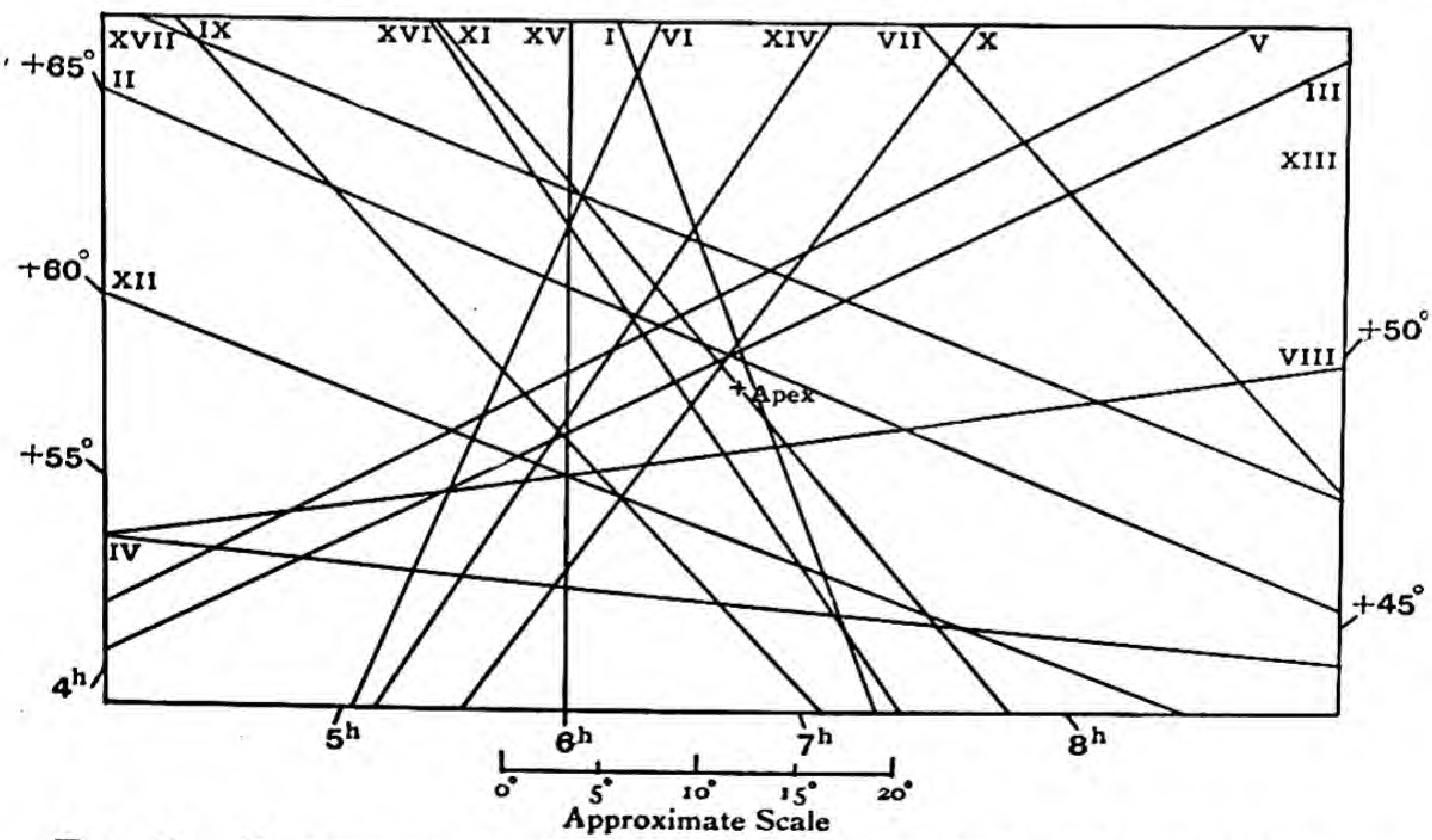


FIG. 11.—Convergence of the Directions of Drift II. from the 17 Regions.

on the sky and a photograph taken of the part of the sky where they converge; the great circles on such a photograph will appear as straight lines. These are shown on the

two diagrams, and the Roman numeral attached to each line indicates the region from which it comes. Each diagram represents an area of the sky measuring about 60° by 30°; this would correspond on the terrestrial globe to a map of Northern Africa from the Congo to the Mediterranean. The apex marked on each diagram is the definitive apex of the drift, determined by a mathematical solution. For one of the drifts, called Drift I., the convergence of the directions is so evident as to need no comment. Owing to the smaller velocity of Drift II., its direction in any region cannot be determined so accurately, and a greater deviation of the great circles must be expected. Having regard to this, the agreement must be considered good, Region VII. being the only one showing important discordance. To appreciate the evidence of this diagram we may make a terrestrial comparison;—if from seventeen points distributed uniformly over the earth tracks (great circles) were drawn, every one of which passed across the Sahara, they might fairly be considered to show strong evidence of convergence; the distribution of the Drift II. directions is quite analogous.

The analysis of the regions gives not only the directions of the two drifts, but also their speeds in terms of a certain unit; and both sets of results may be used in finding definitive positions of the apices towards which the two drifts are moving. The results of a solution by least squares are as follows:

	Drift I.			Drift II.		
	Apex.			Apex.		
	R.A.	Dec.	Speed.	R.A.	Dec.	Speed.
10 Equatorial Regions	92°·4	−14°·1	1·507	286°·5	−63°·6	0·869
7 Polar Regions . .	89°·3	−16°·7	1·536	289°·1	−63°·6	0·816
Whole Sphere . .	90°·8	−14°·6	1·516	287°·8	−64°·1*	0·855

* The fact that the declination derived from the whole sphere does not lie between the declinations from the two portions looks paradoxical, but is due to the unequal weights of the determinations of the *Z* component from the two portions.

The speeds are measured in terms of the usual theoretical unit $1/h$.

The drift-speed in any region should (owing to foreshortening) vary as the sine of the angular distance from the drift-apex, being greatest 90° away from the apex and diminishing to zero at the apex and antapex. This progressive decrease as the regions get nearer the apex is well shown in the observed values, and the sine-law is followed with very fair accuracy.

Another fact derived from the analysis is the proportion in which the stars are divided between the two streams; this seems to vary somewhat from region to region, but the mean result is that 59·6 per cent. belong to Drift I. and 40·4 per cent. to Drift II.; that is a proportion of practically 3 : 2.

To sum up, the result of this analysis of the Boss proper motions is to show that the motions can be closely represented if there are two drifts. That which we have called Drift I. moves with a speed of 1·52 units, the other, Drift II., with a speed of 0·86 unit. The first drift contains $\frac{3}{5}$ of the stars and the second drift $\frac{2}{5}$. Their directions are inclined at an angle of 100°.

It will be remembered that these motions are measured relative to the Sun. In Fig. 12, let *SA* and *SB* repre-

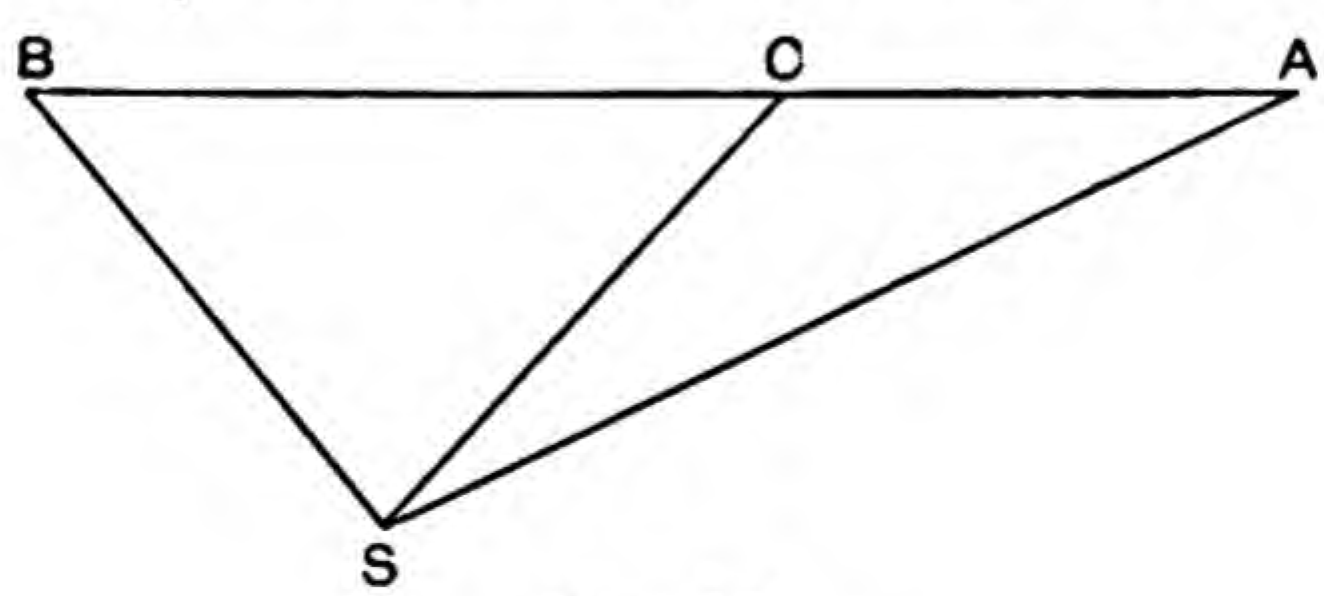


FIG. 12.

sent the drift-velocities, making an angle of 100°. Divide *AB* at *C* so that $AC : CB = 2 : 3$ corresponding to the proportion of stars in the two drifts. Then *SC* represents

the motion of the centroid of all the stars relative to the Sun, and accordingly *CS* represents the solar motion and points towards the solar apex. *AB* and *BA* represent the motion of one drift relative to the other; the points in the sky towards which this line is directed are called the *Vertices*. The positions found from the numbers above are

Vertices	R.A. 94°·2	Dec. +11°·9
	R.A. 274°·2	Dec. −11°·9.

The relative velocity of the two drifts is 1·87 units.

It is a remarkable fact that the vertices fall exactly in the galactic plane, so that the relative motion of the two drifts is exactly parallel to the galactic plane.

The solar motion *CS* found from the same numbers is 0·91 towards the

Solar Apex . . .	R.A. 267°·3	Dec. +36°·4

This may be compared with Prof. Boss's determination from the same catalogue by the ordinary method [4]:

Solar Apex . . .	R.A. 270°·5	Dec. +34°·3

The agreement is interesting because the principles of the two determinations are very different; moreover, in Boss's result the magnitudes of the proper motions as well as their directions were used, whereas the analysis on the two-drift theory depends solely on the directions.

Since the speed of the solar motion has also been measured in kilometres per second, this provides an equation for converting our theoretical unit into linear measure. We have 0·91 unit = 19·5 km./sec., whence the theoretical unit $1/h$ is 21 kilometres per second. We can thus, if we wish, convert any of the velocities previously given into kilometres per second.

It will be seen from Fig. 12 that the direction of motion of Drift I., *SA*, is inclined at a comparatively small angle to the parallactic motion, *SC*. But that the directions

are clearly distinct may be appreciated by referring back to Fig. 10. The solar antapex actually falls just outside that diagram, so it is clear that the convergence is not towards the solar antapex but towards a different apex at the point indicated.

When referred to the centroid of the stars instead of to the Sun, the motions *CA* and *CB* of the two drifts are seen to be opposite to one another. It is perhaps not easy to realise that the inclination of the two stream-motions is a purely relative phenomenon depending on the point of reference chosen; but this is the case. If we divest our minds of all standards of rest and contemplate simply two objects in space—two star-systems—all that can be said is that they are moving towards or away from or through one another along a certain line. The distinction between meeting directly or obliquely disappears. It is clear that this line joining the vertices must be a very important and fundamental axis in the distribution of stellar motions. It is an axis of symmetry, along which there is a strong tendency for the stars to stream in one direction or the other. It is this point of view that has led to an alternative mode of representing the phenomenon of star-streaming, the ellipsoidal theory of K. Schwarzschild.[5]

We have, hitherto, analysed the stars into two separate systems, which move, one in one direction, the other in the opposite direction, along the line of symmetry; but Schwarzschild has pointed out that this separation is not essential in accounting for the observed motions. It is sufficient to suppose that there is a greater mobility of the stars in directions parallel to this axis than in the perpendicular directions. The distinction is a little elusive, when it is looked into closely. It may be illustrated by an analogy. Consider the ships on a river. One observer states that there are two systems of ships moving in opposite directions, namely, those homeward

bound and those outward bound; another observer makes the non-committal statement that the ships move generally along the stream (up or down) rather than across it. This is a not unfair parallel to the points of view of the two-drift and ellipsoidal hypotheses. The distinction is a small one and it is found that the two hypotheses express very nearly the same law of stellar velocities; but by the aid of different mathematical functions. This will be shown more fully in the mathematical discussion in the next chapter. Meanwhile we may sum up in the words of F. W. Dyson [6]: "The dual character of Kapteyn's system should not be unduly emphasised. Division of the stars into two groups was incidental to the analysis employed, but the essential result was the increase in the peculiar velocities of stars towards one special direction and its opposite. It is this same feature, and not the spheroidal character of the distribution, which is the essential of Schwarzschild's representation."

The phenomenon of star-streams (by which we mean the tendency to stream in two favoured directions, which both the two-drift and ellipsoidal theories agree in admitting) is shown very definitely in all the collections of proper motions that are available for discussion. Very careful attention has been given to the question whether it could possibly be spurious, and due to unsuspected systematic errors in the measured motions.[7] It is not difficult for the investigator to satisfy himself that such an explanation is quite out of the question, but it is not so easy to give the evidence in a compact form. Happily we are able to give one piece of evidence which seems absolutely conclusive. F. W. Dyson [8] has made an investigation of the stars (1924 in number) with proper motions exceeding 20″ a century. In this case we are not dealing with small quantities just perceptible with refined measurements, but

with large movements easily distinguished and checked. These large motions show the same phenomenon that has been described for the smaller motions of Boss's catalogue. In fact, the two streams are shown more prominently when we leave out the smaller motions. This does not mean that the more distant stars are less affected by star-streaming than the near stars; it is easy to show that for stars at a constant distance the small proper motions must

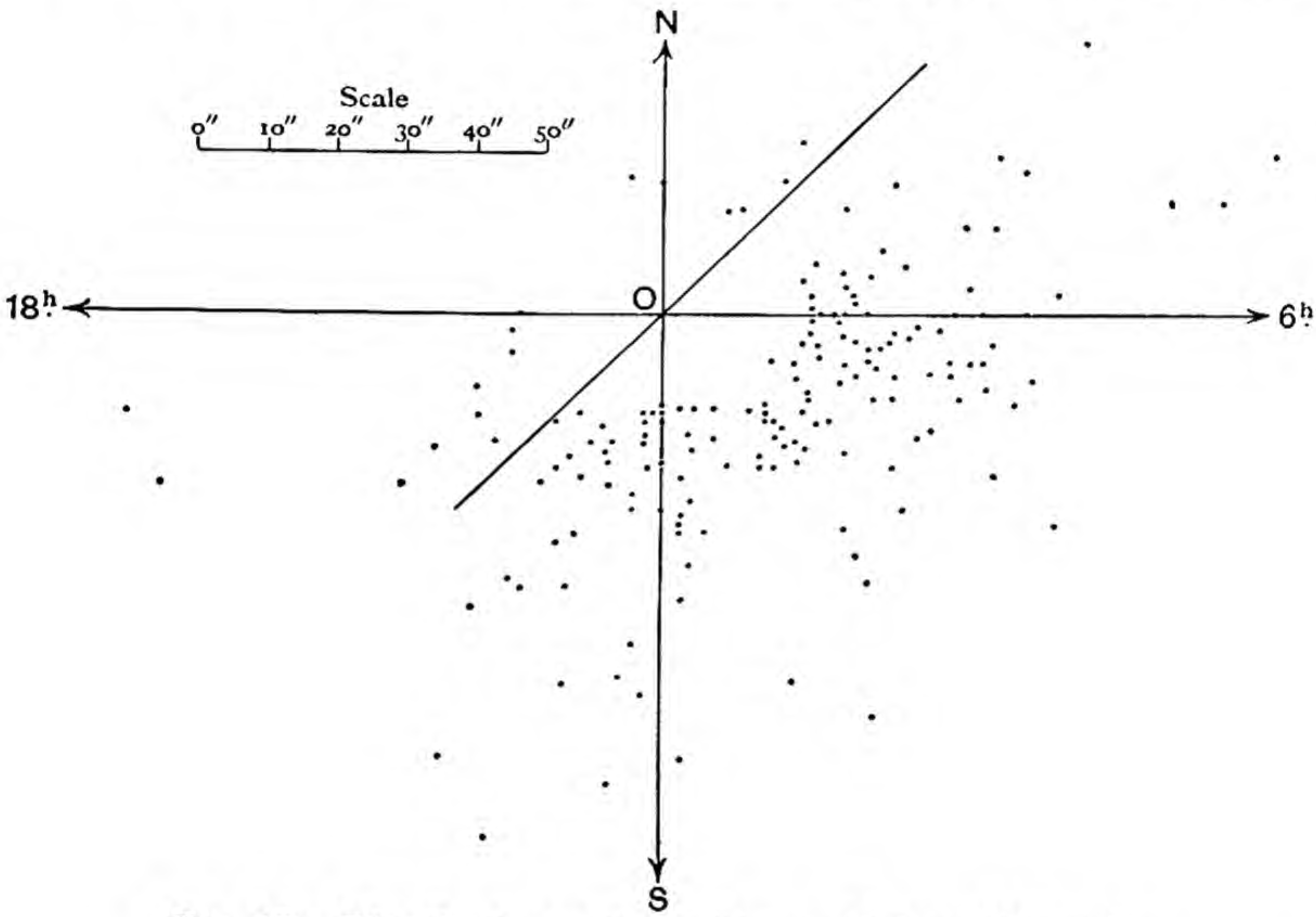


FIG. 13.—Distribution of Large Proper Motions (Dyson).

necessarily be distributed more uniformly in position angle than the large motions, and the enhancement of the streaming when the small motions are removed is due to this cause.

The diagram Fig. 13 is taken from Dyson's paper; it refers to the area R.A. 10^h to 14^h, Dec. $-30°$ to $+30°$. The motion of each star is represented by a dot, the displacement of the dot from the origin representing a century's motion on the scale indicated. The blank space round the origin is, of course, due to the omission of all motions less than 20″; we can imagine it to be filled with an extremely

dense distribution of dots. It is clear that the distribution shown in the diagram represents a double streaming approximately along the axes towards 6^h and *S.* No single stream from the origin could scatter the dots as they actually are. Although the general displacement is towards the solar antapex (*i.e.*, towards the lower right corner), this is accompanied by an extreme elongation of the distribution in a direction almost at right angles. Thus the phenomenon of two streams is well shown in the largest, and proportionately most trustworthy, motions that are known, so that it requires no particular delicacy of observation to detect it.

Radial Motions

Doubtless the most satisfactory confirmation of this phenomenon found in the transverse motions of the stars would be an independent detection of the same phenomenon in the spectroscopically measured radial velocities. Although great progress has been made in the determination and publication of radial velocities, the stage has scarcely yet been reached when a satisfactory discussion of this question is possible. We shall see that the results at present available are quite in agreement with the two-stream hypothesis, and afford a valuable confirmation of it in a general way; but a larger amount of data is required before we can see how precise is the agreement between the two kinds of observations.

In the transverse motions the two streams were detected by considering the stars in a limited area of the sky; there was no need to go outside a single area, except at a later stage when it was desired to show that the different parts of the sky were concordant. But with the radial velocities, we can learn nothing of star-streaming from a single region; that is the drawback of a one-dimensional projection compared with a two-dimensional. To pass from one area to another involves questions of stellar dis-

tribution, which complicate the problem. In particular it is necessary to pay attention to spectral type. It is well known that the early type stars are more numerous near the galaxy than elsewhere; as these have on the average smaller residual motions than the later types, there will be a tendency for the radial motions near the galactic plane to be smaller than near the poles. But evidence of star-streaming should be looked for in a tendency for the residual radial velocities to be greater near the vertices (which are in the galactic plane) than elsewhere. The two effects are opposed, and there is a danger that they will mask one another.

By treating the different types separately the difficulty is avoided, but in that case the data become rather meagre. For Type A the results have been worked out by Campbell [9] who gives the following table:

TABLE 9.

Average Residual Velocities. Type A.

Galactic Latitude.	Distance from Kapteyn's Vertices.		
	0—30°	30—60°	60—90°
0—30°	$15{\cdot}9_{33}$	$10{\cdot}3_{33}$	$11{\cdot}7_{35}$ km. per sec.
30—60°	—	$11{\cdot}2_{46}$	$7{\cdot}6_{36}$
60—90°	—	—	$9{\cdot}3_{29}$
Mean . .	$15{\cdot}9_{33}$	$10{\cdot}8_{79}$	$9{\cdot}5_{100}$

The suffixes show the number of stars.

The increased velocity in the neighbourhood of the vertices seems to be plainly marked, and it is in fair quantitative agreement with the results from the transverse velocities.* A division of the results according to galactic

* There seems to have been some misunderstanding on this point. It is considered mathematically in Chapter VII.

latitude is given, showing that the progressive increase is not dependent upon that.

The radial velocities of the later types of spectrum have not yet been discussed.

General Characteristics of the Members of the Two Streams

We now turn to the question whether there is any physical difference between the stars of the two streams. Are they of the same magnitude and spectral type on the average? And are they distributed at the same distance from the Sun, and in the same proportions in all parts of the sky? It is not possible on the two-drift theory definitely to assign each star to its proper drift; all that can be said is that, of the stars moving in a specified direction, a certain proportion belong to Drift I. and the rest to Drift II. There are, however, directions in which the separation is nearly complete, and, by confining ourselves to these, we may pick out a sample of stars of which 90 per cent. or more belong to Drift I., and another sample equally representative of Drift II. Our samples are thus not absolutely pure, but they are sufficient for testing whether there is any physical difference between the members of the two drifts.

By thus separating the drifts as far as possible, the following table has been constructed to compare the magnitudes of the stars. The stars are those of Boss's catalogue (stars of Type B being omitted). Approximately equal samples of each drift have been taken from each of ten regions; the results for the five polar and five equatorial regions are shown separately in Table 10.

The last two columns of the table agree closely; the slight excess of very bright stars in Drift I. may be noticed, but it seems to be of an accidental character.

TABLE 10.

Magnitudes of Stars of the Two Streams.

Magnitude.	Polar Regions. I. II. V. VI. & VII.		Equatorial Regions. VIII. IX. XII. XIII. & XVII.		Total.	
	Drift I.	Drift II.	Drift I.	Drift II.	Drift I.	Drift II.
0·0—2·9	16	7	6	4	22	11
3·0—3·9	17	10	10	12	27	22
4·0—4·9	46	52	38	38	84	90
5·0—5·4	50	52	39	52	89	104
5·4—5·9	99	100	78	68	177	168
6·0—6·4	75	72	75	79	150	151
6·5—6·9	50	59	57	41	107	100
7·0—	44	51	52	49	96	100
Variable	7	2	0	2	7	4
Total .	404	405	355	345	759	750

The investigation may be extended to somewhat fainter stars by taking the Groombridge proper motions.[10] Samples taken from these give :—

	Number of stars of magnitudes								
	0—3·9.	4·0—4·9.	5·0—5·9.	6·0—6·9.	7·0—7·4.	7·5—7·9.	8·0—8·4.	8·5—8·9.	Total.
Drift I.	16	29	86	171	136	108	104	51	701
Drift II.	3	23	81	169	125	113	132	61	707

The excess of very bright stars is again noticed, but this is to some extent a repetition of the stars which occurred in the last table and is not fresh evidence. Further the Type B stars were not excluded in forming this table; these form a considerable proportion of the bright stars, and their motions are known to be peculiar. At the other end of the table an excess of faint stars in Drift II. is shown, but it is not very decided. This may be spurious, being the effect of a greater accidental error in determining the directions of motion of the faintest stars, which tends to increase falsely the number assigned to the slower-moving drift.

The main conclusion from the two tables is that there is no important difference in the magnitudes of the stars constituting the two drifts. On the other hand there is possibly some significance in the fact that discussions of the faint stars, such as the Groombridge and Carrington stars, have given a higher proportion belonging to Drift II. than the discussions of brighter stars, such as those of Bradley and Boss.

The same course may be pursued with regard to spectral types, though, in view of the known differences in the amount of the individual motions of late and early type stars, such a treatment is unsatisfactory, and the interpretation of the result is ambiguous. We give, however, results for four regions of the Groombridge stars[11] :—

TABLE 11.

Spectra of Stars of the Two Streams.

Region.	Drift I.			Drift II.		
	No. of stars. Type I.	No. of stars. Type II.	Percentage of Type II.	No. of stars. Type I.	No. of stars. Type II.	Percentage of Type II.
A	61	66	52	36	70	66
B	95	35	27	61	45	42
C	61	23	27	16	16	50
G	58	39	40	41	39	49

In each case the percentage of Type II. (later type) stars is higher in Drift II. than in Drift I. But some caution is needed in interpreting this table. It may be that the result is due to the elements of the star-stream motions differing from one type to another. We cannot tell whether the distribution of spectra differs from one drift to the other, or whether the drift-motions differ from one spectral type to another. This matter is still undecided; but, knowing at least

one remarkable relation between type and motion, we cannot ignore the second alternative.

A safe way of stating the conclusion is, that stars of early and late types are found in both streams, but that there is a somewhat higher proportion of late types among stars moving in the direction of Drift II. than of Drift I.

Distances of the Two Streams

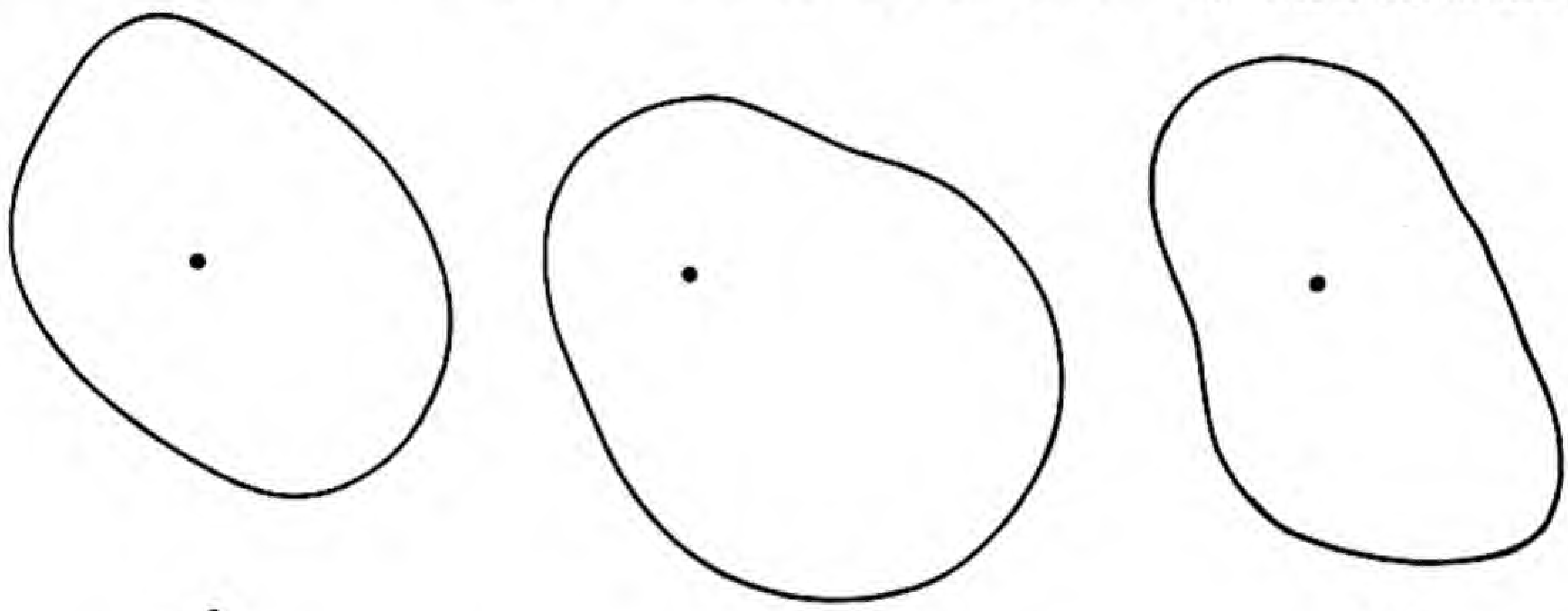
It is most important to determine whether the two streams are actually intermingled in space. It might, for example, be suggested that one of the streams consists of a cluster of stars surrounding the Sun, which moves relatively to the background of stars constituting the other stream. The absence of any appreciable correlation between magnitude and drift renders such an explanation rather improbable, for it would be expected that the stars of the background would be fainter on the average than those of the nearer swarm. The question can, however be treated more definitely by using the magnitude of the proper motions to measure the distances of the two drifts. Hitherto we have only made use of the directions of the motions without reference to the amount; we must now bring the latter element into consideration.

Let d_1 and d_2 be the respective mean distances * of the two drifts; if these are known, the theory (set forth in the next chapter) enables us to calculate the mean proper motion of stars moving in any direction. Take for instance Fig. 14, which refers to a region of the Groombridge catalogue; † the curves are drawn so that the radius vector in any direction measures the mean proper motion in the corresponding direction. The velocities of the drifts and the numbers of stars

* Unless otherwise specified the *mean distance* of a system of stars means the harmonic mean distance, or distance corresponding to the mean parallax.

† This is the "restricted Region G," *Monthly Notices*, Vol. 67, p. 52.

belonging to each have first been found by the usual method; we can then draw the theoretical mean proper motion curves for any assumed values of d_1 and d_2. Two such curves are shown, viz., for $d_1 = d_2$ and for $d_1 = \frac{1}{2} d_2$. The first curve A has a slight bi-lobed tendency, that is to say, there are two directions in which the radius vector is a maximum; but it will be seen that the mean proper motion curve is not a very sensitive indicator of the presence of two streams. That does not matter for our present purpose. The upper part of the curve arises mainly from stars of Stream II., the lower part from Stream I. If we decrease the average distance of Stream I. and increase



A B C

FIG. 14.—Mean Proper Motion Curves.

A. Theoretical Distribution.--Two Drifts at the same Mean Distance.
B. Theoretical Distribution.—Second Drift at twice the Distance of First Drift.
C. Observed Distribution.

that of Stream II. the lower part of the curve will expand and the upper part shrink. This is what has happened in curve B.

The remaining curve represents the results of observation. We have purposely selected a region containing a large number of stars (767), so that the observed curve is fairly smooth; but a mean proper motion is nearly always liable to large accidental fluctuations, and we must not expect a very close agreement with theory. It will be noticed that C is much more definitely bi-lobed

than the theoretical curves; the two streams are more prominent than was anticipated. This phenomenon is found not only in this particular example, but in all the divisions of the Groombridge motions. It suggests that our two-drift analysis has not succeeded in giving a complete account of the facts. (The ellipsoidal hypothesis fails equally in this respect.) The precise significance of the failure has not been made out; we cannot do more than call attention to an outstanding difference.

The difference in shape of the curves *A* and *C* renders a comparison somewhat difficult, but it will be recognised that the proportion of the two lobes is not very different, pointing to approximately equal distances for the two drifts. There is no such magnification of one lobe at the expense of the other as is illustrated in curve *B*.

The values of d_1 and d_2 can be obtained by a rigorous mathematical solution by least squares. The results for this region (G) and six others are given in Table 12.[12]

TABLE 12.

Mean Parallaxes of the Two Drifts.

Region.	Limits of Region.		Drift I.		Drift II.	
	Dec. N.	R.A.	$\frac{1}{hd_1}$	Probable Error.	$\frac{1}{hd_2}$	Probable Error.
	°	h	″	″	″	″
A	70—90	0—24	2·96	±0·07	3·41	±0·13
B	38—70	22— 2	2·45	0·07	2·40	0·11
C	38—70	2— 6	2·39	0·08	2·65	0·12
D	38—70	6—10	3·35	0·12	3·23	0·21
E	38—70	10—14	3·65	0·14	4·78	0·29
F	38—70	14—18	3·74	0·23	4·15	0·31
G	38—70	18—22	2·77	0·16	2·55	0·18

The quantities $1/hd_1$ and $1/hd_2$ are the mean parallaxes multiplied by a factor the value of which is probably about 450. As, however, some large proper motions have been

excluded (the same proportion from each drift), the absolute parallaxes have no definite significance. It is the ratios that are of importance.

The table shows that in each region the two drifts are at nearly the same mean distance. In only one case, Region E, is the difference at all significant, and even there the ratio, about 4 : 3, necessitates a very considerable intermixture of the two sets of stars. Moreover, Region E contains fewer stars than any other, and the results are in consequence uncertain. It is thus necessary to regard the two star-streams as thoroughly intermingled systems, and to throw aside any hypothesis which regards them as passing one behind the other in the same line of sight.

The table also shows a variation in mean distance from region to region, which is greater than the variation from Drift I. to Drift II. This variation is found to follow the galactic latitude of the stars, and is due to the fact that we see a larger number of more distant stars the nearer we get to the plane of the Milky Way. As both drifts show this progressive change, these distant stars must belong to both drifts impartially.

A similar conclusion is derived from the more recent and accurate proper motions of Boss's catalogue. In order to obtain a large enough number of stars we have thrown together Regions VIII., XII., XIII. and XVII. of the previous division, and considered the large region consisting of two antipodal areas each about 70° square. This region is one of high galactic latitude, so that the proper motions are comparatively large; and, its centre being nearly 90° from both Apex and Vertex, the large extent of the area is less harmful than it would be in other parts of the sky. This region contains 1122 stars.

In Table 13 the second column gives the number of stars moving in each 10° sector. The third column gives the mean proper motion of these stars; in order to smooth out minor

irregularities, these means have been taken for overlapping 30° sectors. The fourth column gives the calculated mean proper motion, based on the known velocities and

TABLE 13.

Mean Proper Motions of a Region of Boss's Catalogue

(Centre of Region, R.A. 0^h, Dec. 0°).

Direction.	No. of Stars.	Mean Centennial P.M.	
		Observed.	Calculated.
°		"	"
0	94	12·73	12·61
10	90	12·20	12·61
20	88	11·15	12·54
30	66	10·27	12·32
40	79	10·56	11·45
50	50	8·92	10·51
60	41	9·49	9·35
70	34	8·82	8·99
80	33	10·00	8·91
90	30	11·19	9·13
100	25	10·85	9·35
110	33	11·24	9·57
120	34	8·74	9·57
130	36	8·98	9·35
140	50	9·09	9·35
150	25	9·56	8·99
160	32	8·90	8·55
170	13	7·05	8·12
180	10		
190	6		
200	7		
210	4		
220	1		
230	4	5·98	6·23
240	6		
250	9		
260	10		
270	8		
280	6		
290	4	6·42	6·09
300	20	7·33	6·88
310	10	7·83	8·05
320	14	9·17	9·20
330	26	10·40	10·15
340	44	11·27	11·16
350	80	12·46	12·03

relative proportions of the drifts, and on the assumption that they are at the same mean distance. As the number of stars moving in directions between 175° and 285° is too small to give trustworthy separate mean proper motions, these have been combined to give one mean.

The corresponding polar diagrams are given in Fig. 15. The agreement is very fair; but, as in the previous case,

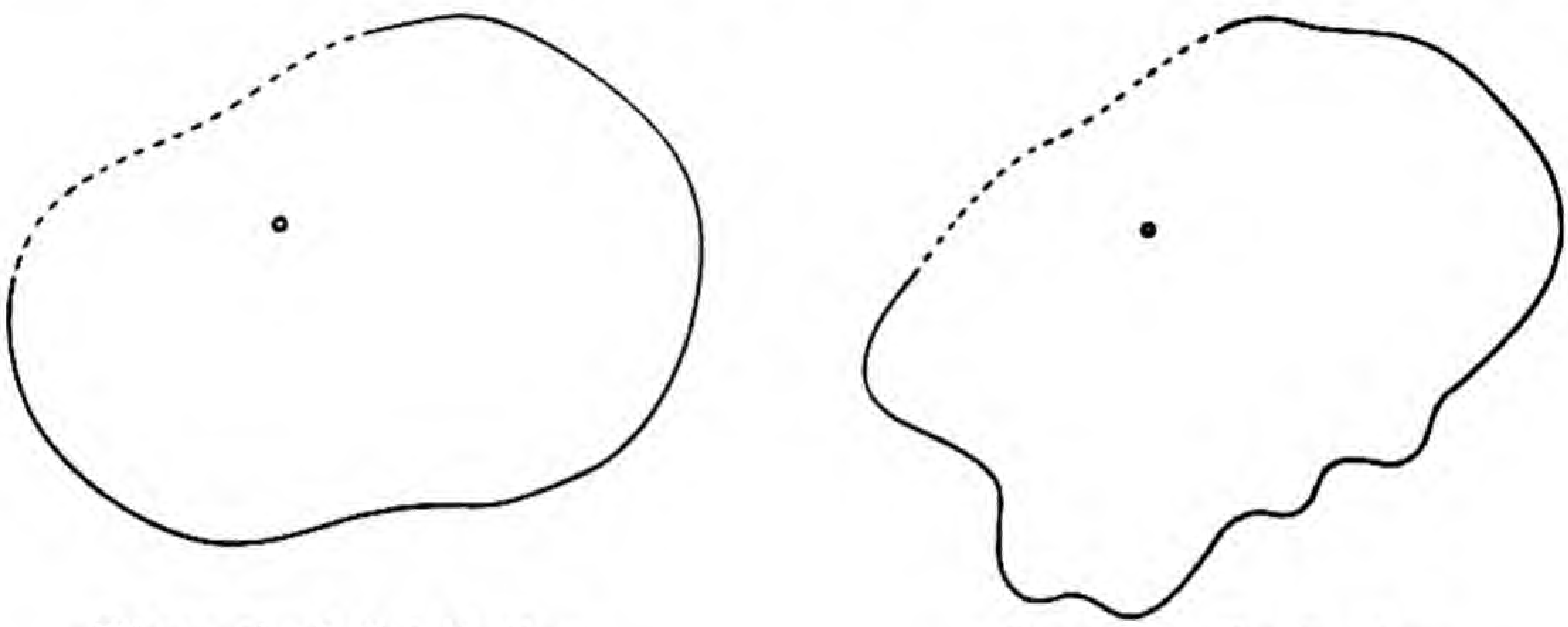


FIG. 15.—Mean Proper Motion Curve (Region of Boss's Catalogue).

the two drifts appear more sharply in the observed curve than in the theoretical one. It is clear that our assumption of equal distances for the two drifts cannot be far wrong. A rigorous solution by least squares leads to the results:—

For Drift I. . . . . $\frac{1}{hd} = 6''{\cdot}94 \pm 0''{\cdot}10$ per century

,, Drift II. . . . . $\frac{1}{hd} = 7''{\cdot}38 \pm 0''{\cdot}17$,,

Or, adopting the value of the unit $1/h$ already found, viz., 21 km. per sec., the mean parallaxes are

Drift I. . . . . $0''{\cdot}0156 \pm 0''{\cdot}00023$
Drift II. . . . . $0''{\cdot}0166 \pm 0''{\cdot}00038$

On the question whether the proportion of mixture remains the same in all parts of the sky, conflicting views have been held. We are not concerned with local irregularities, but with any general tendency of

one drift to prevail over a hemisphere or belt of the sky. It seems to be agreed that there is no systematic connection between the numbers and galactic latitude. Table 14 taken from the analysis of Boss's catalogue shows this clearly. The individual irregularities may

TABLE 14.

Division of the Stars between the Drifts.

Boss's Region.	No. of Stars.	Ratio Drift II. : Drift I.	Galactic Latitude.
			°
III.	304	0·60	1
X.	356	0·66	1
VII.	448	0·87	9
II.	354	0·48	12
XVI.	311	0·69	14
XV.	285	0·71	17
I.	371	0·60	27
IX.	275	0·75	29
XI.	365	0·42	31
IV.	294	0·87	33
XVII.	245	0·77	37
VIII.	308	0·67	44
VI.	294	0·52	46
XIV.	259	0·68	48
XII.	342	0·64	61
V.	274	1·03	66
XIII.	237	0·59	78

be partly real and partly the result of insufficient data or errors in the proper motions; but there is no systematic progression. As already explained, these regions consist each of two antipodal areas, and hence the results do not allow us to test whether there is a difference between two opposite hemispheres of the sky. According to S. S. Hough and J. Halm [13] there is a considerable difference. From a discussion of the radial velocities of stars they deduced that the Drift II. stars were concentrated in the hemisphere towards the point R.A. 324°, Dec. – 12° ; the range of density was not explicitly determined but it was evidently considerable, the Drift

II. stars being relatively two or three times as numerous at one pole compared with the other. This result depended, at least partly, on an apparent general dilatation of the stellar system or excess of positive radial velocities compared with negative. At the present time the excess is more generally attributed to a systematic error in the radial velocities of certain types of stars, possibly attributable to a pressure displacement of the spectral lines. The result cannot, therefore, be regarded as trustworthy in itself. Analysis of the angular motions, however, confirmed the general conclusion.

The same authors,[14] from an examination of the Bradley proper motions, found a maximum density of Drift II. at approximately 0^h R.A. and in a southern declination, possibly the south galactic pole; they showed further that this inequality of distribution would fully account for certain anomalies found by Newcomb in his discussion of the precessional constant, viz., a difference in the results from the right ascensions and declinations, and a residual twelve-hour term $-0''\cdot50 \cos\delta \cos 2\alpha$ in the mean proper motions in right ascension. In his latest researches Halm has adopted a more complicated three-drift hypothesis; his results (based now on the Boss proper motions) still indicate a very marked excess of Drift II. stars towards about 22^h in general agreement with his former conclusions.

To summarise this discussion of the distribution and characters of the stars of the two streams, we may say that on the whole the mixture is remarkably complete. Such differences as are found are difficult to interpret with certainty, being liable to be influenced by small systematic errors in the proper motions; and in most cases it is scarcely possible as yet to discriminate between a difference in the proportion of the drifts and a difference in their velocities. An excess of later spectral type stars in Drift II., and a relative excess of

Drift II. stars in the southern galactic hemisphere, are the most important differences found; but there are indications that this crude interpretation of statistical results is not an adequate description of the complex distribution of motions in different parts of the stellar system and among different classes of stars.

The recognition of two star-streams may be expected to throw some light on the discrepancies in earlier determinations of the solar motion. Of these one of the most remarkable is that due to H. Kobold,[15] who in fact was led by it to a partial recognition of the systematic motions of the stars. Using Bessel's method, Kobold found for the solar apex the position R.A. 269°, Dec. −3°, differing by at least 35° from the position generally accepted. We have shown (p. 83) that Bessel's method depends very essentially on the individual motions of the stars being distributed nearly in accordance with the law of errors. As that is now known to be untrue, it is not surprising that the point found is very different from the real apex. It is easy to see that, but for the solar motion, Bessel's method would give a very sensitive determination of the vertex; and it is really much better adapted for determining the vertex than the apex. The point found by Kobold is indeed quite close to the vertex found on the two-drift or ellipsoidal hypotheses, the presence of the solar motion having produced a deviation of only a few degrees. There is another way of looking at the matter. In applying Bessel's method to determine the solar motion, only the line joining the apex and antapex is found, and there is nothing to indicate which end is the apex. If there are two drifts, we might expect that the line determined by Kobold would be a sort of weighted mean between the corresponding lines of the two drifts. This is actually the case. But naturally Kobold's line is in the acute angle between the drift-axes, whereas the solar

motion is the other mean lying in the obtuse angle between them.

It was customary in early determinations of the solar apex to consider separately the proper motions between various limits of size. It was always found that the declination of the apex decreased as the magnitude of the proper motions increased. This is easily explained on the two-drift theory. Since Drift I. has much the greater velocity relative to the Sun, the stars belonging to it have on an average greater proper motions than the Drift II. stars. Thus the larger the motions discussed, the greater will be the proportion of Drift I. stars, and the nearer will be the resulting apex to the Drift I. apex.

In like manner the higher declination of the solar apex of the later type stars may be accounted for by the increased proportion of Drift II. in the later types. The higher declination of the apex for faint stars, and the higher proportion of faint stars in Drift II. (both somewhat doubtful deductions from the observations) would also correspond.

Three-drift Hypothesis.

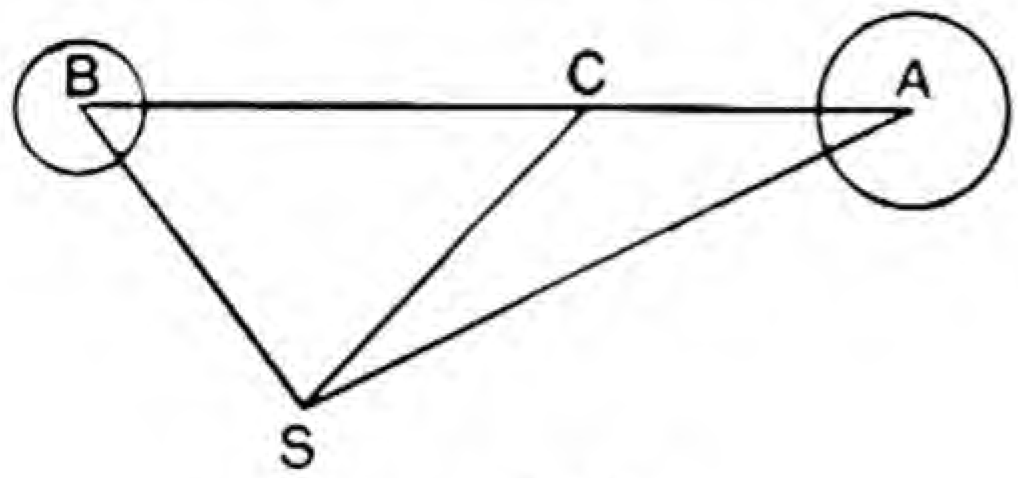
From what has been said as to the relations of the two-drift and ellipsoidal hypotheses, it will be understood that we do not regard the analysis as giving more than an approximation to the actual law of stellar motions. Its importance is that it takes account of what is clearly the most striking feature of the distribution. It has been found, however, that one systematic deviation can be detected, of which neither of these hypotheses is able to take account; we are indeed ready to take a step forward towards a second approximation. In comparing the observed distribution with that calculated on the two-drift or ellipsoidal theories, it is found that there is always some excess of stars moving towards the solar antapex. This is shown in the diagrams of the Boss

proper motions (and, in fact, the Groombridge motions) by a bulge of the curve roughly in the antapex direction. This was attributed by the writer [16] to a small third stream, of much less importance than the two great drifts. Halm [17] has shown, however, that it is better to re-analyse the motions on the assumption of three drifts. According to his representation, there is a third drift, Drift O, which is practically at rest in space, and is thus intermediate between the two original drifts. Naturally in introducing the third drift into the analysis some of the stars originally included in Drift I. and Drift II. are regrouped with Drift O, and the elements of the two former become somewhat modified. This is especially so with Drift I., which moves more nearly in the same direction as Drift O, and it is mainly at its expense that the new drift is formed.

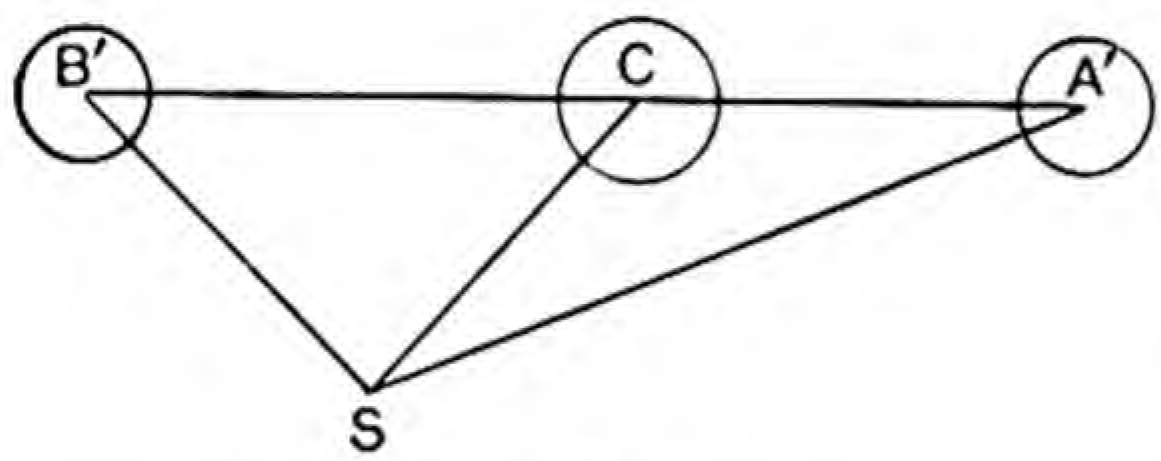
That Halm's interpretation of the phenomenon is the right one may be seen by looking at the diagrams for Regions XIV. and XVI. (Fig. 9, IX and X). In most cases Drift I. and Drift O overlap so much that they appear almost as one drift; only a little extra bulge of the curve towards the antapex betrays the duality. Regions XIV. and XVI. are the two places where their directions become more open, and in these the three distinct streams of about equal importance are plainly manifest. Remembering that it is only in these two regions that we could expect Drift I. and Drift O to be really separate, the evidence for the three-drift hypothesis becomes very strong. Moreover the third drift was postulated by Halm, not so much to explain these peculiarities, but because it seemed needed to reconcile results derived from different parts of the celestial sphere.

The hypotheses of two-drifts and of three-drifts, or we should rather say the two successive approximations, may be compared thus: In Fig 16, the line *CS*, which represents the solar motion, is the same in each case. In the first diagram we have Stream I. with $\frac{3}{5}$ of the stars

and Stream II. with $\frac{2}{5}$ of the stars, their motions relative to the Sun being SA and SB. Since C is the centre of mass of the whole, $CB : CA = 2 : 3$, and CA, CB are the motions of the two streams freed from the solar motion. In the second diagram we have regrouped the stars into three roughly equal streams, the motions of which relative to the Sun are SA', SB', SC; their absolute motions are CA', CB' and zero, and here CA' is approximately equal to CB'. Clearly also $A'B'$ must be greater than AB, for in removing the slower moving members of A and B to form a new group at C, we increase the relative mean velocity of the remainder.



Two Drifts



Three Drifts

FIG. 16.—Comparison of Two-drift and Three-drift Theories.

The third drift has little or no motion relatively to the mean of the stars; it may be considered to be practically at rest in space. This characteristic is a well-known feature of the stars of the Orion type, which are usually removed from the data, in making investigations of the two star-streams, because they are found not to participate in the drift motions.* It is natural to associate Drift O with the Orion stars as sharing the same peculiarity of motion; but it is not quite clear how close is the relation between them.

* The Orion stars were removed from the writer's investigation of Boss's Preliminary General Catalogue, but not for this reason, which was in fact unknown at that time. Their tendency to form vague moving clusters was the main objection; further, as the motions of those not included in the clusters is extremely small and, therefore, the direction of the motion is inaccurately determined, it was thought that the value of the results would be improved by their removal (*Monthly Notices*, vol. 71, p. 40).

The elements of the three drifts determined by Halm from the Boss proper motions are :—

	Apex. R.A.	Apex. Dec.	Speed.
Drift I.	90°	0°	$hV=1{\cdot}5$
Drift II.	270°	−49°	$hV=0{\cdot}9$
Drift O	90°	−36°	$h'V=1{\cdot}5$

From the analogy between the Drift O and the Orion type stars, Halm was led to assume that their internal motions are smaller than those of the other drifts ; thus

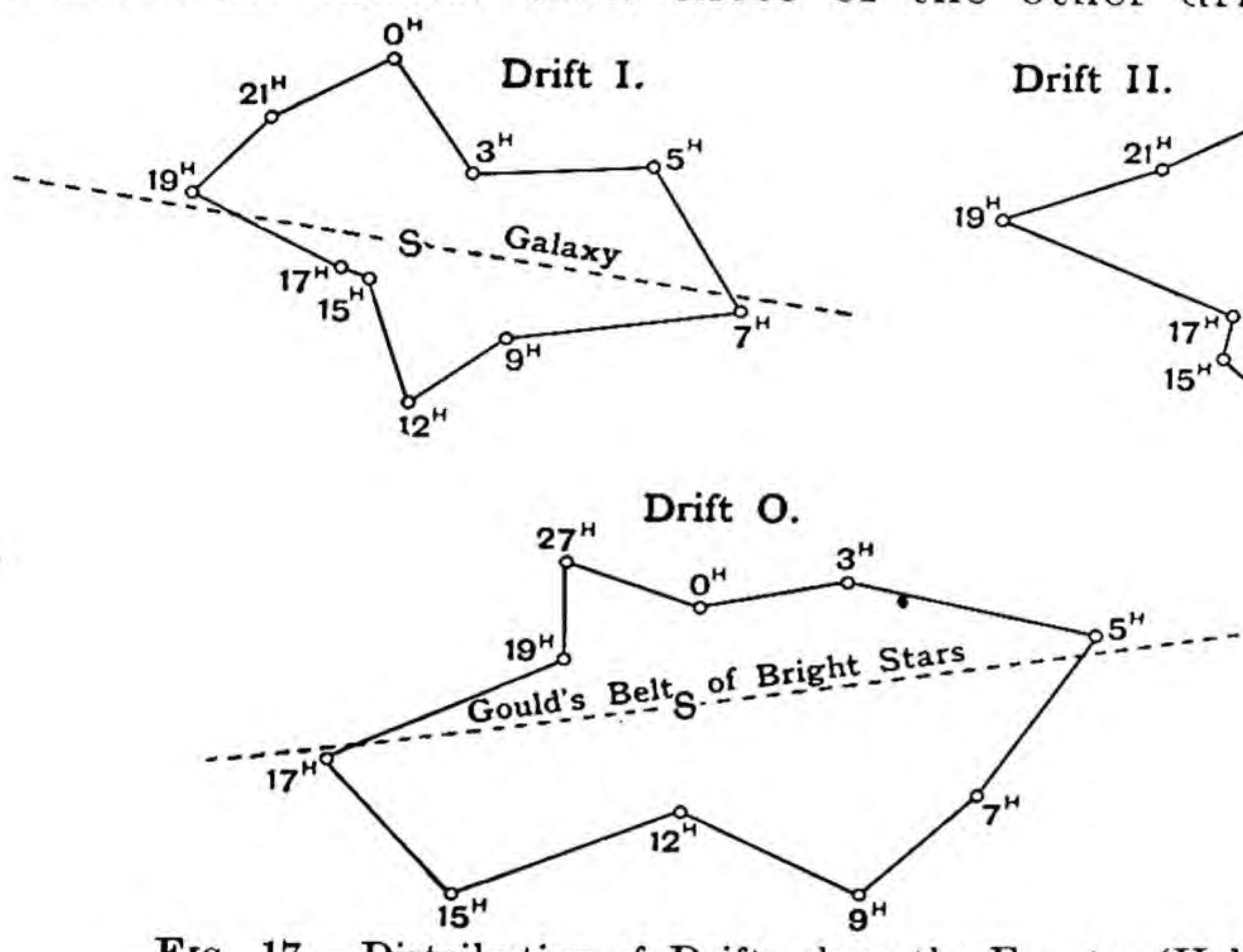


FIG. 17.—Distribution of Drifts along the Equator (Halm).

the speed given above is measured in terms of a different unit h'.

The additional number of constants to be determined in the three-drift analysis makes the results from individual regions very uncertain. Great variations in the relative proportions of the stars in the drifts were found ; these are partly accidental owing to the uncertainty of the analysis, but are doubtless also partly real. The diagram (Fig. 17) shows the distribution of the stars along the equator in different right ascensions, the radius drawn from *S* showing the number of stars belonging to the drift. The apparent relation between

Drift O and Gould's belt of bright stars is probably due to the fact that many of the brightest stars belong to the Orion type.

Following the attitude we have adopted with regard to the other two hypotheses, we may be content to regard the three-drift theory as only an analytical summary of the distribution of stellar motions, without any hypothesis as to the physical existence of three separate systems. There is, however, an important property of Drift O, which seems to make it something more than a mathematical abstraction. We have seen that all the stars of the Orion type appear to belong to it, and not to the two other drifts. It is true that in addition it contains stars of the other types, which are not in any way distinctive; but that it should contain the whole of one spectral class seems to show that it corresponds to some real physical system, and places it on a somewhat different footing from the two older drifts, for which we have as yet failed to find any definite characteristic apart from motion.

SUMMARY OF DETERMINATIONS OF THE CONSTANTS OF THE TWO STREAMS.

True Vertex.

Ref. No.	Catalogue or Data used.	Investigator.	Vertex. R.A.	Vertex. Dec.	Hypothesis.
			°	°	
1	Auwers-Bradley . . .	Kapteyn . . .	91	+13	Two-stream.
2	,, . . .	Rudolph . .	96	+ 7	Ellipsoidal.
3	,, . . .	Hough and Halm	90	+ 8	Two-drift.
4	Groombridge	Eddington . . .	95	+ 3	Two-drift.
5	,, ...	Schwarzschild .	93	+ 6	Ellipsoidal.
6	Boss	Eddington . . .	94	+12	Two-drift.
7	,, ...	Charlier	103	+19	Generalised Ellipsoidal.
8	Zodiacal	Eddington . . .	109	+ 6	Two-drift.
9	Large Proper Motions .	Dyson	88	+21	Two-drift.
10	,, .	Beljawsky . . .	86	+24	Ellipsoidal.
11	Radial Velocities . . .	Hough and Halm	88	+27	Two-drift.
12	Boss	Boss	not given		Two-stream.
13	Faint stars ($7^m{\cdot}0 - 13^m$).	Comstock . . .	87	+28	Ellipsoidal.

With regard to these it may be remarked that the method used in No. 7 gives greatest weight to the nearest stars, so that Nos. 7, 9, 10 refer mainly to the stars closest to our system. No. 10 is somewhat tentative, as the analysis used is not rigorously applicable to stars selected on account of large proper motions. No. 8 is probably affected by a systematic error in the data used. No. 11 would probably be revised, if account were taken of the systematic error now believed to exist in the determinations of radial velocity of the Orion type stars.

Apices of the Two Drifts.

Ref. No.	Catalogue.	Investigator.	Drift I. R.A.	Drift I. Dec.	Drift II. R.A.	Drift II. Dec.
			°	°	°	°
1	Auwers-Bradley . . .	Kapteyn . . .	85	−11	260	−48
3	,, . . .	Hough and Halm	87	−13	276	−41
4	Groombridge	Eddington . . .	90	−19	292	−58
6	Boss	Eddington . . .	91	−15	288	−64
12	,, ...	Boss	96	− 8	290	−54
9	Large Proper Motions .	Dyson	93	− 7	246	−64

For the velocities of the two drifts relative to the sun the results are :—

No. 3	ratio 3 : 2
,, 4	1·7 and 0·5
,, 6	1·52 ,, 0·86
,, 9	ratio 3 : 2

For the ratio of the minor axis to the major axis of the Schwarzschild ellipsoid, the determinations are :—

No. 2	0·56
,, 5	0·63
,, 7	0·51
,, 10	0·47
,, 13	0·62

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CHAPTER VII

THE TWO STAR-STREAMS—MATHEMATICAL THEORY

In the preceding chapter we have described the principal results of researches on the two star-streams. We shall now consider the analytical methods used in the investigations.

Two-drift Hypothesis

Consider a region of the sky, sufficiently small to be treated as plane, and consider the motions of the stars projected on it. Suppose that there is a drift of stars, *i.e.* a system, in which the *motûs peculiares* are haphazard, but which is moving as a whole relatively to the Sun. We take, as the mathematical equivalent of haphazard, a distribution of velocities according to Maxwell's Law, so that the number of stars with individual linear motions between (u,v) and $(u+du, v+dv)$ is

$$\frac{Nh^2}{\pi}e^{-h^2(u^2+v^2)}du\,dv.$$

In justification of this it may be pointed out that Maxwell's is the only law for which the frequency is the same for all directions, and at the same time there is no correlation between the x and y components of velocity. There are other laws, which make the motions random in direction; but for them the expectation of the value of a component velocity u will differ according to the value of

the other component v; for instance, with the law $e^{-h\sqrt{(u^2+v^2)}}$ a large component v is likely to be accompanied by a large component u. Now we are not at the moment concerned with what law stellar motions are likely to follow; that is a dynamical problem. We are rather choosing a standard of comparison with which to compare the actual distribution of motions, and that standard ought to be the simplest possible. It is by no means unlikely that large values of v may be correlated with large values of u; but, if that should be the case, we ought to discover it as an explicit deviation from the simpler assumption of no correlation rather than conceal it in our initial formulæ; for it is an interesting result that has to be accounted for. In seeking for the unknown law of stellar velocities, we are at liberty to adopt any standard of comparison that we please, but there is clearly a special propriety in taking Maxwell's Law, as it is the nearest possible approach to an absolutely chaotic state of motion.

In the frequency-law

$$\frac{Nh^2}{\pi}e^{-h^2(u^2+v^2)}\,du\,dv.$$

N is the total number of stars considered and h is a constant depending on the average *motus peculiaris*. It is related to the mean speed (in three dimensions) Ω by the equation

$$\Omega = \frac{1}{h}\sqrt{\frac{4}{\pi}}$$

Let

V = the velocity of the drift, taken to be along the axis of x;
r = resultant velocity of a star;
θ = position angle, or inclination to Ox, of the resultant velocity

(The velocities are all to be taken in linear, not angular measure.)

Then

$$u^2 + v^2 = r^2 + V^2 - 2Vr\cos\theta$$
$$du\,dv = r\,dr\,d\theta.$$

Thus the number of resultant motions between position angles θ and $\theta + d\theta$ is

$$N\frac{h^2}{\pi}d\theta\int_0^\infty e^{-h^2(r^2+V^2-2Vr\cos\theta)}rdr.$$

Setting

$$x = h(r - V\cos\theta)$$
$$\tau = hV\cos\theta$$

the number is

$$\frac{N}{\pi}e^{-h^2V^2}d\theta\, e^{\tau^2}\int_{-\tau}^\infty e^{-x^2}(x+\tau)dx$$

$$= \frac{N}{\pi}e^{-h^2V^2}d\theta\left\{\tfrac{1}{2}+\tau e^{\tau^2}\int_{-\tau}^\infty e^{-x^2}dx\right\}$$

Writing

$$f(\tau) = \frac{2}{\sqrt{\pi}}\left\{\tfrac{1}{2} + \tau e^{\tau^2}\int_{-\tau}^\infty e^{-x^2}dx\right\}$$

the following table gives the values of $\log f(\tau)$

TABLE 15.

The Function $f(\tau)$.

τ.	$\log f(\tau)$.	τ.	$\log f(\tau)$.	τ.	$\log f(\tau)$.	τ.	$\log f(\tau)$.
−1·2	$\bar{1}$·0411	−0·3	$\bar{1}$·5363	0·5	0·1876	1·3	1·1520
−1·1	$\bar{1}$·0874	−0·2	$\bar{1}$·6046	0·6	0·2886	1·4	1·3003
−1·0	$\bar{1}$·1355	−0·1	$\bar{1}$·6763	0·7	0·3947	1·5	1·4555
−0·9	$\bar{1}$·1856	0·0	$\bar{1}$·7514	0·8	0·5061	1·6	1·6177
−0·8	$\bar{1}$·2378	0·1	$\bar{1}$·8303	0·9	0·6232	1·7	1·7871
−0·7	$\bar{1}$·2923	0·2	$\bar{1}$·9131	1·0	0·7461	1·8	1·9637
−0·6	$\bar{1}$·3493	0·3	0·0001	1·1	0·8751	1·9	2·1478
−0·5	$\bar{1}$·4088	0·4	0·0916	1·2	1·0103	2·0	2·3393
−0·4	$\bar{1}$·4711						

For a single drift the number of stars moving in any direction θ is proportional to $f\ (hV\cos\theta)$; and the equation to the theoretical single-drift curves discussed on p. 88 is

$$r \propto f(hV\cos\theta).$$

The usual method of analysis is to compound two such curves pointing in different directions, adjusting the

various parameters by trial and error until a satisfactory approximation to the observations is obtained.

A mathematical method of determining the double-drift formula

$$r = a_1 f(hV_1 \cos \overline{\theta - \theta_1}) + a_2 f(hV_2 \cos \overline{\theta - \theta_2}),$$

which best represents the observations, without recourse to trial and error, has been given.[1] It worked quite satisfactorily for the Groombridge regions, where the stars were very numerous ; it is not, however, to be recommended. A mechanical method, which automatically gives some sort of answer, whether the distribution really corresponds to two drifts or not, is not so discriminating as the simpler synthetic process.

Ellipsoidal Hypothesis.

It has been seen that the principal fact of the phenomenon of star-streaming is the greater mobility of the stars along a certain line than in the perpendicular directions. K. Schwarzschild represents this mobility by assuming that the individual motions are distributed according to the modified Maxwellian law

$$e^{-k^2u^2 - h^2(v^2+w^2)}$$

where, k being less than h, the u components of velocity are on the average greater than the v and w components.

In two dimensions, let the number of stars with individual motions between (u, v) and $(u+du, v+dv)$ be

$$\frac{Nhk}{\pi} e^{-k^2u^2 - h^2v^2} du\, dv.$$

And let the components of the parallactic motion of the whole system be (U, V). The parallactic motion is not in general along the axis of greatest mobility Ox.

As before, let r, θ be the magnitude and direction of the resultant velocity of a star. Then

$$k^2u^2 + h^2v^2 = k^2(r\cos\theta - U)^2 + h^2(r\sin\theta - V)^2,$$
$$du\, dv = r\, dr\, d\theta.$$

The number of stars moving in directions between θ and $\theta+d\theta$

$$=\frac{Nhk}{\pi}d\theta\int_0^\infty rdr\, e^{-r^2(k^2\cos^2\theta+h^2\sin^2\theta)+2r(k^2U\cos\theta+h^2V\sin\theta)-k^2U^2-h^2V^2}.$$

Setting

$$p=k^2\cos^2\theta+h^2\sin^2\theta,$$

$$\xi=\frac{k^2U\cos\theta+h^2V\sin\theta}{\sqrt{p}}.$$

$$x=r\sqrt{p}-\xi$$

The number becomes

$$\frac{Nhk}{\pi}\,d\theta\, e^{-k^2U^2-h^2V^2}\int_0^\infty e^{-pr^2+2\xi r\sqrt{p}}\,rdr$$

$$=\frac{Nhk}{\pi}\,d\theta\, e^{-k^2U^2-h^2V^2}\frac{e^{\xi^2}}{p}\int_{-\xi}^\infty e^{-x^2}(x+\xi)dx.$$

The integral leads to the same function f as before; and the number of stars moving in any direction is proportional to

$$\frac{1}{p}f(\xi).$$

A little consideration shows that the polar curve $r=\dfrac{1}{p}f(\xi)$ will closely resemble a two-drift curve. ξ is a maximum near the direction of the parallactic motion (U, V) and a minimum in the opposite direction; and the same is true for $f(\xi)$. This factor alone would give a curve not very different from a single drift curve. But the factor $\dfrac{1}{p}$ corresponds to an ellipse with its major axis along Ox. It distorts the approximately single-drift curve, pinching it along Oy and extending it along Ox, with the result that a bi-lobed curve is usually obtained.

The method of determining the constants of an ellipsoidal distribution to suit the observations may be briefly noticed.* If we consider the stars moving in a direction θ and in the opposite direction $180°+\theta$, p is the same for both directions, and ξ simply changes sign. Thus the

* I have made slight alterations in the procedure given by Schwarzschild in order to conserve the close correspondence with the two-drift analysis.

ratio of the numbers of stars moving in these directions gives $\frac{f(\xi)}{f(-\xi)}$. From the table of $\log f$, we construct the following table (the logarithm is given as being more convenient for interpolation):—

TABLE 16.

Auxiliary Function for the Ellipsoidal Theory.

ξ.	$\log \frac{f(\xi)}{f(-\xi)}$.	ξ.	$\log \frac{f(\xi)}{f(-\xi)}$.
0·0	0·000	0·5	0·779
0·1	0·154	0·6	0·939
0·2	0·309	0·7	1·102
0·3	0·464	0·8	1·268
0·4	0·620		

This enables ξ to be found from the observations, and, when it is known, p is given by

$$\text{number of stars} = \frac{1}{p} f(\xi).$$

If now we take radii $r_1 = \frac{1}{\sqrt{p}}$ and $r_2 = \xi\sqrt{p}$ in the direction θ, r_1 will trace out the ellipse

$$k^2 r_1^2 \cos^2\theta + h^2 r_1^2 \sin^2\theta = 1,$$

and r_2 the straight line

$$r_2 = k^2 U \cos\theta + h^2 V \sin\theta.$$

By drawing the best ellipse and straight line through the respective loci, k^2, h^2, U and V are readily found; also the direction of greatest mobility, which is the major axis of the above ellipse, is determined.

A very elegant direct method of arriving at the values of these constants has also been given by Schwarzschild [2]; it appears, however, to be open to the same objection as the automatic method of determining the constants of the two star-streams. If these methods are used at all, it is very necessary to examine afterwards how closely the

solutions represent the original observations. I have sometimes found them to be very misleading.

The ellipsoidal hypothesis has been applied with great success by Schwarzschild to the analysis of the proper motions of the Groombridge catalogue. As an illustration we may take the region R.A. 14^h to 18^h, Dec. $+38^\circ$ to $+70^\circ$ already considered on the two-drift theory (p. 90). The comparison of the two hypotheses with the observations is given in Table 17. In only two lines do the two representations differ from one another by more than a unit.

TABLE 17.

Comparison of Ellipsoidal and Two-drift Hypotheses with Observations.

Direction.	Number of Stars.			Direction.	Number of Stars.		
	Observed.	Ellipsoidal.	Two Drift.		Observed.	Ellipsoidal.	Two Drift.
5	4	5	6	205	21	22	22
15	5	6	7	215	27	25	26
25	6	7	8	225	29	26	27
35	9	9	10	235	26	27	26
45	10	11	11	245	19	23	22
55	14	12	12	255	17	18	18
65	14	13	12	265	12	14	14
75	14	14	13	275	11	10	10
85	13	13	13	285	11	8	8
95	12	13	12	295	8	6	6
105	10	12	13	305	7	5	5
115	11	11	12	315	6	5	4
125	10	10	11	325	6	4	5
135	10	10	9	335	5	4	5
145	7	10	9	345	5	5	6
155	9	11	9	355	4	5	6
165	9	12	11				
175	14	14	12				
185	14	16	15				
195	16	19	19				

Whilst the two hypotheses can yield closely similar distributions of motion as regards direction, it is conceivable that if the magnitudes of the motions were taken into account the resemblance might fail. But it is not

difficult to see that the two laws express distributions of linear velocities very similar in most respects, though by the aid of different mathematical functions, provided that the numbers of stars in the two drifts are practically equal. Schwarzschild's method is somewhat analogous to replacing two equal intersecting spheres by a spheroid. If we have two equal drifts, with velocities $+V$ and $-V$ referred to the centre of mass of the whole, the frequency of a velocity (u, v) is proportional to

$$e^{-h^2((u-V)^2+v^2)}+e^{-h^2((u+V)^2+v^2)},$$

or to

$$e^{-h^2(u^2+v^2)}\cosh(2h^2Vu).$$

The ellipsoidal law may be written

$$e^{-h^2(u^2+v^2)}\cdot e^{(h^2-k^2)u^2}.$$

The difference in the two laws is thus determined by the difference between cosh au and $e^{\beta u^2}$, functions which have a considerable general resemblance.

As both laws give the same distribution of the v components, we may confine attention to the u components. As a typical example for comparing the laws (corresponding approximately with what is actually observed), we will take $hV=0{\cdot}8, \dfrac{k}{h}=0{\cdot}58, \dfrac{1}{h}=20$ km. per sec.

TABLE 18.

Comparison of Two-drift and Ellipsoidal Hypotheses.

Component u. Km. per sec.	Frequency.		Difference Ellipsoidal − Two-drift.
	Two-drift Hypothesis.	Ellipsoidal Hypothesis.	
0	1·055	1·160	+0·105
10	1·099	1·066	−0·033
20	1·000	0·828	−0·172
30	0·618	0·544	−0·074
40	0·237	0·302	+0·065
50	0·056	0·141	+0·085
60	0·008	0·056	+0·048
70	0·001	0·019	+0·018

The two curves are shown in Fig. 18.

Whilst there is a marked general resemblance, the differences are not altogether negligible. In particular the ellipsoidal law gives a considerably greater number of large velocities; it seems probable that in this respect it is better fitted to the observations. The two-drift law gives a defect of both very small and very large motions, as compared with the simple error law; this is sometimes expressed by saying that it has a *negative excess.*

It may further be noticed that, although in the example given the frequency of the two-drift distribution

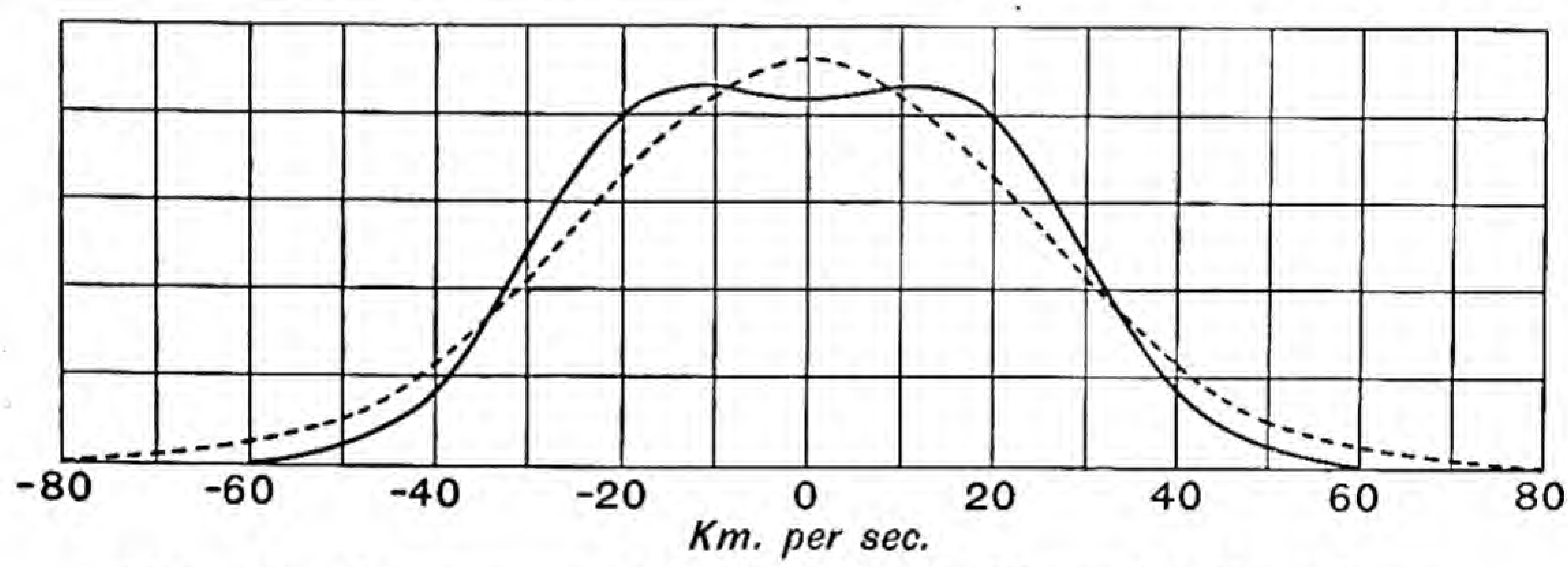

FIG. 18.—Comparison of Two-drift and Ellipsoidal Hypotheses.

Two-drift—full curve; Ellipsoidal—dotted curve.

makes a slight dip at the origin $u=0$, for rather smaller values of hV this dip disappears, and the distribution actually agrees with the ellipsoidal distribution in having a maximum at the origin.[3]

When the restriction that the two drifts have equal numbers of stars is removed, the ellipsoidal hypothesis cannot approximate to the two-drift hypothesis so closely. There is no longer a fore-and-aft symmetry, so that the ellipse is an unsuitable figure to represent the frequency. The two-drift theory, having one additional disposable constant, is now able to give a considerably better representation of the observations. We have seen that the Groombridge proper motions can be represented by

two drifts with approximately equal numbers of stars; these can be replaced by an ellipsoidal distribution with practically the same precision. The Boss proper motions, on the other hand, require a mixture of the drifts in the proportion of about 3 : 2; the ellipsoidal hypothesis cannot be adapted to this skewness and accordingly fails to represent these observations. For this reason it has not been possible to analyse these more recent proper motions on Schwarzschild's theory.* On the other hand the main teaching of that theory remains, viz., that the dissection into two drifts may be only a mathematical procedure, and that it is possible to regard the distribution of velocities as one whole.

Combination of Results from Different Regions of the Sky.

In the case of the two-drift theory the procedure is very simple. Let X_1, Y_1, Z_1 be the components of the velocity in space of one of the drifts, measured in the usual unit $1/h$; v_1 and θ_1, its velocity and position angle,† determined for a region whose centre is at (a, δ). We have for each region equations of condition

$$v_1 \sin \theta_1 = - X_1 \sin a + Y_1 \cos a,$$
$$v_1 \cos \theta_1 = - X_1 \cos a \sin \delta - Y \sin a \sin \delta + Z_1 \cos \delta,$$

from which X_1, Y_1, Z_1 can be found.

On the ellipsoidal hypothesis the projected solar motion is found for each region, and the results can be combined in the same way. But the combination of the ellipsoidal constants is a more complex problem.

It will be useful to take the general ellipsoid with three unequal axes, although in ordinary applications a spheroid

* C. V. L. Charlier has developed a generalisation of the ellipsoidal theory which permits the skewness to be taken into account, but his method assumes some knowledge of the distribution of the stars in space.

† The position angle is here measured from the meridian.

only is considered. Referred to any rectangular axes, let the *velocity-ellipsoid* be

$$au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv = 1,$$

so that the number of stars with individual velocities between (u, v, w) and $(u+du, v+dv, w+dw)$ is proportional to

$$e^{-(au^2+bv^2+cw^2+2fvw+2gwu+2huv)}\,du\,dv\,dw.$$

Take the w direction to be the line of sight. To obtain the distribution of the projected velocities (u, v) we must integrate the above expression with respect to w from $-\infty$ to $+\infty$. The result is

$$exp - \left\{au^2 + bv^2 + 2huv - \frac{(fv+gu)^2}{c}\right\}.\,du\,dv\int_{-\infty}^{\infty} exp - c\left(w + \frac{fv+gu}{c}\right)^2.\,dw.$$

The integral in this expression is a constant and equal to $\sqrt{\frac{\pi}{c}}$.

Thus the projected velocities correspond to a *velocity-ellipse*

$$au^2 + bv^2 + 2huv - \frac{(fv+gu)^2}{c} = 1.$$

Now this ellipse is the right-section of the cylinder parallel to the w-axis, which passes through the intersection of the ellipsoid

$$au^2 + bv^2 + cw^2 + 2fvw + 2gwu + 2huv = 1,$$

and the plane

$$w = -\frac{fv + gu}{c}.$$

The latter is the diametral plane conjugate to the w-axis, and consequently the cylinder is the enveloping cylinder.

Thus the velocity-ellipse for any region is simply the *outline* of the velocity-ellipsoid viewed from an infinite distance in the corresponding direction. The outline must, of course, not be confused with the cross-section which is a different ellipse.

Now let the velocity-ellipsoid, transformed to its principal axes, be

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{w^2}{c^2} = 1,$$

and the line of sight be in the direction (l, m, n). The lengths of the axes of the enveloping cylinder (*i.e.*, of the velocity-ellipse) are given by

$$\frac{l^2}{a^2 - r^2} + \frac{m^2}{b^2 - r^2} + \frac{n^2}{c^2 - r^2} = 0,$$

and the direction ratios of these axes are then

$$\frac{l}{a^2 - r^2} : \frac{m}{b^2 - r^2} : \frac{n}{c^2 - r^2}.$$

It may be noted that there will be four points of the sky for which the velocity-ellipse will be a circle, and the projected motions will be haphazard. When the ellipsoid is a spheroid these points coalesce with the two extremities of the axis, in other words with the vertices.

The general case of the ellipsoid with three unequal axes is of considerable interest, because it enables us to allow for the possibility that the motions may have a special relation to the plane of the Milky Way as well as to the axis of star-streaming. The fact that stellar motions have some tendency to be parallel to the Milky Way was pointed out by Kobold, and in recent years by the investigators of radial velocities. Since the stellar system is strongly flattened towards this plane, the result seemed a very natural relation between motion and distribution. But the star-stream investigations have shown that the main tendency is not towards a general parallelism to the Milky Way but a parallelism to a certain direction in it. Whether there is any residual relation to the Milky Way not covered by this, is a matter of some interest. We could test it by making an analysis on the basis of a velocity-ellipsoid with three unequal axes, and noticing whether the two smaller axes turn out

to be equal to one another. The difficulty is that, as already stated, the ellipsoidal hypothesis does not give a satisfactory representation of Boss's proper motions, which would naturally be used for such a test. It seems, however, fairly certain that the deviation from a spheroid must be very slight. The evidence from the radial motions (Table 9) points in the same direction.

Taking the velocity-ellipsoid to be a prolate spheroid

$$k_1^2u^2 + h_1^2(v^2 + w^2) = 1$$

and the velocity-ellipse for a region, with centre at an angular distance χ from the vertex, to be

$$k^2u^2 + h^2v^2 = 1,$$

then, since the velocity-ellipse is the apparent outline of the ellipsoid, we have

$$\frac{1}{k^2} = \frac{\cos^2\chi}{k_1^2} + \frac{\sin^2\chi}{h_1^2}$$

and

$$h = h_1,$$

and hence

$$\left(\frac{h^2}{k^2} - 1\right) = \left(\frac{h_1^2}{k_1^2} - 1\right)\cos^2\chi.$$

Thus the minor axis $\frac{1}{h}$ of the velocity-ellipse is the same throughout the sky; and the last equation expresses the variation of the major axis.

Also the major axis is directed along the great circle to the vertex.

Mean Proper Motions.

These have been used in the previous chapter to determine the mean distances of the two drifts. It would have been possible to start with the mean proper motions in the different directions, instead of the simple frequency, as data, for the purpose of exhibiting and analysing the star-stream phenomena. But, besides being much less sensitive, there is the objection noticed

in Airy's method of finding the solar motion, that an average motion is likely to depend mainly on a few specially large values, and is very much subject to accidental fluctuations. To use the frequency of proper motions leads to much smoother results; and further, as we avoid giving excessive weight to the nearest stars, the results should be more representative of the stars as a whole. It is therefore better to reserve the mean proper motions for obtaining new information not deducible from the frequencies.

We found that on the ellipsoidal hypothesis the number of stars moving in directions between θ and $\theta+d\theta$ was proportional to

$$\frac{e^{\xi^2}}{p}\int_{-\xi}^{\infty} e^{-x^2}(x+\xi)dx.$$

Hence for the mean value of x

$$\bar{x}=\int_{-\xi}^{\infty} e^{-x^2}(x^2+x\xi)dx \div \int_{-\xi}^{\infty} e^{-x^2}(x+\xi)dx$$

The numerator of this expression is equal to

$$-\tfrac{1}{2}\int_{-\xi}^{\infty}(x+\xi)\,d(e^{-x^2})$$

$$=-\tfrac{1}{2}\left[(x+\xi)\,e^{-x^2}\right]_{-\xi}^{\infty}+\tfrac{1}{2}\int_{-\xi}^{\infty} e^{-x^2}\,dx.$$

The integrated part vanishes at both limits. We thus obtain

$$\bar{x}=\frac{\tfrac{1}{2}e^{\xi^2}\int_{-\xi}^{\infty} e^{-x^2}dx}{\tfrac{1}{2}+\xi e^{\xi^2}\int_{-\xi}^{\infty} e^{-x^2}dx}$$

$$=\frac{f(\xi)-\dfrac{1}{\sqrt{\pi}}}{2\xi f(\xi)}$$

But $\bar{x}=\bar{r}\sqrt{p}-\xi$.
Hence if

$$g(\xi)=\frac{f(\xi)-\dfrac{1}{\sqrt{\pi}}}{2\xi f(\xi)}+\xi,$$

the mean linear motion in any direction is

$$\bar{r}=\frac{1}{\sqrt{p}}g(\xi).$$

For a simple drift $\sqrt{p}$ reduces to h, and ξ to $hV \cos \theta$, so that the mean linear motion is

$$\bar{r} = \frac{1}{h} g(hV \cos \theta).$$

The values of g are given in Table 19.

TABLE 19.

The Function $g(\tau)$.

τ.	$g(\tau)$.	τ.	$g(\tau)$.	τ.	$g(\tau)$.
-1·0	0·565	-0·1	0·845	0·8	1·315
-0·9	0·589	0·0	0·886	0·9	1·381
-0·8	0·614	0·1	0·930	1·0	1·449
-0·7	0·641	0·2	0·977	1·1	1·520
-0·6	0·670	0·3	1·027	1·2	1·594
-0·5	0·701	0·4	1·079	1·3	1·669
-0·4	0·734	0·5	1·134	1·4	1·747
-0·3	0·768	0·6	1·191	1·5	1·827
-0·2	0·805	0·7	1·252	1·6	1·908

For determining the distances of the two drifts, equations of condition are formed as follows :—
Let

$\bar{r}$ be the mean proper motion in the direction θ.
d_1, d_2 the unknown mean distances of the stars of the two drifts (*i.e.* distances corresponding to the mean parallaxes)
n_1, n_2 the numbers of stars of the two drifts moving in a direction θ. These have been determined by the previous analysis of the directions of motion.
V_1, θ_1; V_2, θ_2 the velocities and directions of the drift motions

Then

$$\overline{n_1+n_2}\,\bar{r} = n_1 g(hV_1 \cos \overline{\theta-\theta_1}) \frac{1}{hd_1} + n_2 g(hV_2 \cos \overline{\theta-\theta_2}) \frac{1}{hd_2}.$$

Forming these equations of condition for successive values of θ, we can determine $\frac{1}{hd_1}$ and $\frac{1}{hd_2}$ by a least-squares solution.

On the ellipsoidal hypothesis there is only the one unknown d, the mean distance of the stars, to deal with. It can be determined by the equations of condition

$$\text{mean proper motion in direction } \theta = \frac{1}{d} \cdot \frac{1}{\sqrt{p}} g(\xi).$$

in Airy's method of finding the solar motion, that an average motion is likely to depend mainly on a few specially large values, and is very much subject to accidental fluctuations. To use the frequency of proper motions leads to much smoother results; and further, as we avoid giving excessive weight to the nearest stars, the results should be more representative of the stars as a whole. It is therefore better to reserve the mean proper motions for obtaining new information not deducible from the frequencies.

We found that on the ellipsoidal hypothesis the number of stars moving in directions between θ and $\theta + d\theta$ was proportional to

$$\frac{e^{\xi^2}}{p}\int_{-\xi}^{\infty} e^{-x^2}(x+\xi)dx.$$

Hence for the mean value of x

$$\bar{x} = \int_{-\xi}^{\infty} e^{-x^2}(x^2 + x\xi)dx \div \int_{-\xi}^{\infty} e^{-x^2}(x+\xi)dx$$

The numerator of this expression is equal to

$$-\tfrac{1}{2}\int_{-\xi}^{\infty}(x+\xi)\, d(e^{-x^2})$$

$$= -\tfrac{1}{2}\left[(x+\xi)\, e^{-x^2}\right]_{-\xi}^{\infty} + \tfrac{1}{2}\int_{-\xi}^{\infty} e^{-x^2}\, dx.$$

The integrated part vanishes at both limits. We thus obtain

$$\bar{r} = \frac{\tfrac{1}{2}e^{\xi^2}\int_{-\xi}^{\infty} e^{-x^2}dx}{\tfrac{1}{2} + \xi e^{\xi^2}\int_{-\xi}^{\infty} e^{-x^2}dx}$$

$$= \frac{f(\xi) - \dfrac{1}{\sqrt{\pi}}}{2\xi f(\xi)}$$

But $\bar{x} = \bar{r}\sqrt{p} - \xi$.
Hence if

$$g(\xi) = \frac{f(\xi) - \dfrac{1}{\sqrt{\pi}}}{2\xi f(\xi)} + \xi,$$

the mean linear motion in any direction is

$$\bar{r} = \frac{1}{\sqrt{p}}g(\xi).$$

For a simple drift $\sqrt{p}$ reduces to h, and ξ to $hV\cos\theta$, so that the mean linear motion is

$$\bar{r} = \frac{1}{h} g(hV \cos\theta).$$

The values of g are given in Table 19.

TABLE 19.

The Function $g(\tau)$.

τ.	$g(\tau)$.	τ.	$g(\tau)$.	τ.	$g(\tau)$.
−1·0	0·565	−0·1	0·845	0·8	1·315
−0·9	0·589	0·0	0·886	0·9	1·381
−0·8	0·614	0·1	0·930	1·0	1·449
−0·7	0·641	0·2	0·977	1·1	1·520
−0·6	0·670	0·3	1·027	1·2	1·594
−0·5	0·701	0·4	1·079	1·3	1·669
−0·4	0·734	0·5	1·134	1·4	1·747
−0·3	0·768	0·6	1·191	1·5	1·827
−0·2	0·805	0·7	1·252	1·6	1·908

For determining the distances of the two drifts, equations of condition are formed as follows :—

Let

$\bar{r}$ be the mean proper motion in the direction θ.

d_1, d_2 the unknown mean distances of the stars of the two drifts (*i.e.* distances corresponding to the mean parallaxes)

n_1, n_2 the numbers of stars of the two drifts moving in a direction θ. These have been determined by the previous analysis of the directions of motion.

V_1, θ_1 ; V_2, θ_2 the velocities and directions of the drift motions

Then

$$\overline{n_1+n_2}\,\bar{r} = n_1 g(hV_1 \cos\overline{\theta-\theta_1})\frac{1}{hd_1} + n_2 g(hV_2\cos\overline{\theta-\theta_2})\frac{1}{hd_2}.$$

Forming these equations of condition for successive values of θ, we can determine $\frac{1}{hd_1}$ and $\frac{1}{hd_2}$ by a least-squares solution.

On the ellipsoidal hypothesis there is only the one unknown d, the mean distance of the stars, to deal with. It can be determined by the equations of condition

$$\text{mean proper motion in direction } \theta = \frac{1}{d}\cdot\frac{1}{\sqrt{p}}\, g(\xi).$$

Or the mean proper motions may be used to effect an independent determination of the ellipsoidal constants exactly as the frequency was used. The latter procedure is much simplified by the aid of the following theorem :—

If in the direction θ a radius is taken, which is the geometric mean between the mean proper motion in the direction θ and the mean proper motion in the opposite direction, the radius will trace out the velocity-ellipse (within an extremely small margin of error).

The mean proper motions in the directions θ and $\theta + 180°$ are respectively

$$\frac{1}{\sqrt{p}}g(\xi) \text{ and } \frac{1}{\sqrt{p}}g(-\xi).$$

Now ξ can never be as great as the ratio of the solar motion to the minor axis of the velocity-ellipsoid. Actually 0·5 is about the upper limit, but to allow an ample margin we shall take also 1·0. From Table 19,

$$\begin{array}{lll} \text{for } \xi = 0{\cdot}0 & \sqrt{g(\xi)\,g(-\xi)} = & 0{\cdot}8862 \\ \phantom{\text{for } \xi =} 0{\cdot}5 & & 0{\cdot}8914 \\ \phantom{\text{for } \xi =} 1{\cdot}0 & & 0{\cdot}9049 \end{array}$$

Thus $\sqrt{g(\xi)\,g(-\xi)}$ may be taken to be constant with an error not greater than one in fifty in the most extreme case. The geometric mean of the mean proper motions is then proportional to $\frac{1}{\sqrt{p}}$, which is the radius of the velocity-ellipse in the corresponding direction.

The theorem provides a short method of finding the velocity-ellipse in any part of the sky, provided a fairly large number of observed proper motions are available. It is subject, however, to the drawback that there are generally some directions in which very few stars are moving, so that we have to take the geometric mean of two quantities, one of which is badly determined and the other unnecessarily well determined. For the numerous motions of the Groombridge Catalogue the method proved

satisfactory, and the results agreed closely with those found from the simple frequencies of the proper motions.[4]

TABLE 20.

Groombridge Proper Motions.

Region.	No. of Stars.	Ratio of Axes of Velocity-Ellipse.	
		Mean Proper Motions.	Frequency of Proper Motions.
A	585	0·59	0·59
B	862	0·56	0·58
C	516	0·76	0·70
D	443	0·82	0·81
E	385	0·65	0·72
F	425	0·53	0·61
G	1103	0·66	0·72

RADIAL MOTIONS—TWO-DRIFT HYPOTHESIS.

The development of the formulæ necessary for the study of radial motions will now be considered. The radial differ from the transverse motions in two respects. (1) The transverse motions allow us to compare the motions in two perpendicular directions in *the same region of the sky;* but to learn anything as to the form of the velocity distribution from the radial motions it is necessary to compare the results from *different regions.* This is evidently a disadvantage, for it introduces the complication of the differences between galactic and non-galactic stars, local drifts, and so on. (2) The results come out in linear measure, independent of the distances of the stars.

It will be assumed throughout that the radial velocities have been corrected for the solar motion, and are accordingly referred to the centre of mass of the system.

The effect of the preferential motions in the directions of the two vertices will be that the radial velocities will

be greater on the average near the vertices than in other parts of the sky. This was illustrated in Table 9 (p. 107).

Taking first the two-drift theory, let V_1 and V_2 be the velocities of the two drifts, referred to the centroid of the whole system; a and $1-a$ the proportion of stars in each drift.

Then

$$aV_1 = (1-a)V_2.$$

The mean radial velocity regardless of sign near the vertices is for Drift I.

$$\frac{h}{\sqrt{\pi}}\int_{-V_1}^{\infty} e^{-h^2v^2}(V_1+v)dv - \frac{h}{\sqrt{\pi}}\int_{-\infty}^{-V_1} e^{-h^2v^2}(V_1+v)dv$$
$$= \frac{2hV_1}{\sqrt{\pi}}\int_0^{V_1} e^{-h^2v^2}\,dv + \frac{e^{-h^2V_1^2}}{h\sqrt{\pi}}$$

and the mean radial velocity at right angles to the vertices is

$$\frac{1}{h\sqrt{\pi}}.$$

Thus for the two drifts the average radial velocity at the vertex is to the radial velocity in a region 90° from the vertices in the ratio

$$a\left\{2hV_1\int_0^{hV_1} e^{-x^2}dx + e^{-h^2V_1^2}\right\} + (1-a)\left\{2hV_2\int_0^{hV_2} e^{-x^2}dx + e^{-h^2V_2^2}\right\}$$

If, using the results of the analysis of Boss's Catalogue, we set

$$a = 0{\cdot}6 \qquad 1-a = 0{\cdot}4$$
$$hV_1 = 0{\cdot}75 \qquad hV_2 = 1{\cdot}12,$$

the ratio becomes 1·727

In Table 9 the mean observed ratio was

$$\frac{15{\cdot}9 \text{ km. per sec.}}{9{\cdot}5 \text{ km. per sec.}} = 1{\cdot}68,$$

and, allowing for the large size of the areas considered, the agreement is remarkably exact. But the confirmation is not quite so satisfactory as it appears at first sight, because the stars of Table 9 are of Type A, a type showing the star-streaming very strongly; and there is no doubt that

if Type A alone had been used in the transverse motions higher values of hV_1 and hV_2 would have been obtained. According to H. A. Weersma [5] the drift-velocities for Type A are

$$hV_1 = 0{\cdot}92 \qquad hV_2 = 1{\cdot}37,$$

the probable error being, however, nearly 10 per cent. These numbers lead to the ratio 2·02.

Considering the uncertainty both of the observed ratio and of the drift-constants for Type A, the discordance between 1·68 and 2·02 is not unduly great.

Radial Motions—Ellipsoidal Hypothesis.

Generally speaking, Schwarzschild's ellipsoidal hypothesis is the most convenient for the mathematical discussion of the radial motions. The first problem is to find the distribution of the radial velocities at a particular point of the sky in terms of the axes of the velocity-ellipsoid. It must be noticed that the distribution of the component-velocities in any direction is by no means the same as the distribution of whole velocities in that direction.

Let the velocity-ellipsoid referred to its principal axes be

$$\frac{u^2}{a^2} + \frac{v^2}{b^2} + \frac{w^2}{c^2} = 1,$$

and let the line of sight be in the direction (l, m, n).

Referring the ellipsoid to three conjugate diameters two of which, a', b', are in the plane perpendicular to (l, m, n) the equation can be written

$$\frac{u'^2}{a'^2} + \frac{v'^2}{b'^2} + \frac{w'^2}{c'^2} = 1,$$

and the frequency of the *oblique* velocity components w' is proportional to

$$e^{-w'^2/c'^2}\,dw'.$$

If now V be the *rectangular* velocity component in the

line of sight, p the perpendicular on the tangent plane normal to the line of sight,

$$\frac{V}{w'}=\frac{p}{c'}.$$

Thus the frequency of a component V in the direction l, m, n is proportional to

$$e^{-V^2/p^2}\,dV,$$

that is

$$e^{-V^2/(a^2l^2+b^2m^2+c^2n^2)}\,dV.$$

The fact that the divisor is the perpendicular on the tangent plane is analogous to the two-dimensional result that the velocity-ellipse is the right section of the tangent-cylinder.

To determine the velocity-ellipsoid from a series of measures of radial velocities, suppose first that the observations are approximately uniformly distributed over the sky. By the foregoing paragraph the mean value of V^2 in any part of the sky is proportional to $a^2l^2+b^2m^2+c^2n^2$, or, referred to more general axes, to a homogeneous expression of the second degree in l, m, n, say E.

Form now from the observed data the coefficients

$$\begin{aligned} A &= \Sigma V^2l^2 & F &= \Sigma V^2mn \\ B &= \Sigma V^2m^2 & G &= \Sigma V^2nl \\ C &= \Sigma V^2n^2 & H &= \Sigma V^2lm. \end{aligned}$$

Then

$$\begin{aligned} A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu \\ = \Sigma V^2(l\lambda + m\mu + n\nu)^2 \\ = \Sigma E(l\lambda + m\mu + n\nu)^2. \end{aligned}$$

This last expression is the moment of inertia of the surface $r^2=E$ * about the plane whose normal is (λ, μ, ν).

The surface is not the velocity-ellipsoid, indeed it is not an ellipsoid at all; it is the inverse of the reciprocal

* The mass being supposed to be distributed proportionately to the solid angle, or, more strictly, proportionately to the number of observations of radial velocity.

ellipsoid. But it is evident that it will have the same principal planes as the velocity-ellipsoid. Thus the directions of the axes of the velocity-ellipsoid are those of the momental ellipsoid of this surface, that is of the quadric

$$A\lambda^2 + B\mu^2 + C\nu^2 + 2F\mu\nu + 2G\nu\lambda + 2H\lambda\mu = 1.$$

These are given by the direction-ratios

$$\frac{1}{GH - F(A-k)} : \frac{1}{HF - G(B-k)} : \frac{1}{FG - H(C-k)},$$

where k has in succession the values of the three roots of the discriminating cubic

$$\begin{vmatrix} A-k & H & G \\ H & B-k & F \\ G & F & C-k \end{vmatrix} = 0.$$

Since the surfaces $r^2 = E$ and $r^2 = \sqrt{E}$ have the same principal planes we may use $|V|$ instead of V^2 in forming the coefficients, so that

$$A = \Sigma \, |V| \, l^2, \quad F = \Sigma \, |V| \, mn, \quad \text{etc.}$$

This is probably a preferable procedure, since squaring the velocities exaggerates the effect of a few exceptional velocities; just as in calculating the mean error of a series of observations it is preferable to use the simple mean residual irrespective of sign rather than the mean-square residual.*

When the observed radial velocities are not uniformly distributed over the sky the problem is more complex, but there is no great difficulty in working out the necessary formulæ.

Effect of Observational Errors.

The accidental errors in the determination of the proper motions must tend to equalise the number of the stars moving in the different directions, and to smooth out the

* This is contrary to the advice of most text-books; but it can be shown to be true.

peculiarities of distribution caused by star-streaming. In consequence, the deduced velocities of the two drifts are likely to be too small. An approximate calculation of the amount of this effect may be made by taking the simple case when the stars are all at the same distance from the Sun.

In this case the accidental errors of the proper motions will simply reappear (multiplied by a constant) as the accidental errors of the linear motions. If the true frequency of a component of linear motion u be

$$\frac{h}{\sqrt{\pi}} e^{-h^2u^2}\, du,$$

and the frequency of an error x in it be

$$\frac{k}{\sqrt{\pi}} e^{-k^2x^2}\, dx,$$

the apparent frequency of linear motion u will be

$$du \int_{-\infty}^{\infty} \frac{h}{\sqrt{\pi}} e^{-h^2(u-x)^2} \cdot \frac{k}{\sqrt{\pi}} e^{-k^2x^2}\, dx$$

$$= \frac{du}{\sqrt{\pi}} \cdot \int_{-\infty}^{\infty} \frac{hk}{\sqrt{\pi}} e^{-(h^2+k^2)x^2+2h^2ux-h^2u^2}\, dx$$

$$= \frac{du}{\sqrt{\pi}} e^{-\frac{h^2k^2}{h^2+k^2}u^2} \cdot \frac{hk}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-(h^2+k^2)\left(x-\frac{h^2u}{h^2+k^2}\right)^2}\, dx$$

$$= \frac{hk}{\sqrt{(h^2+k^2)}} \cdot \frac{1}{\sqrt{\pi}} e^{-\frac{h^2k^2}{h^2+k^2}u^2}\, du.$$

The true frequency distribution with the constant h is thus replaced by an apparent distribution with a constant h_1 where

$$\frac{1}{h_1^2} = \frac{1}{h^2} + \frac{1}{k^2}.$$

To apply this formula we may use the determinations of $\frac{1}{hd}$ which have been made for several regions.

Thus taking Region B of Groombridge's Catalogue

$$\frac{1}{hd} = 2''{\cdot}4 \text{ per century.}$$

The probable accidental error of a Groombridge proper motion is about 0″·7 per century.

Therefore

$$\frac{0\cdot477}{k} = \frac{0''\cdot7}{2''\cdot4} \times \frac{1}{h}$$

$$\frac{1}{k} = \frac{0\cdot61}{h}$$

$$\frac{1}{h_1^2} = \frac{1 + (0\cdot61)^2}{h^2}$$

$$\frac{1}{h_1} = \frac{1\cdot17}{h}$$

Thus for Region B the deduced velocities of the drifts need to be increased in the ratio $\frac{7}{6}$.

For the Boss proper motions the correction is not so important. Taking the large region already discussed (p. 116)

$$\frac{1}{hd} = 7''\cdot2 \text{ per century.}$$

The probable error of a Boss proper motion = 0″·55 per century.

$$\frac{0\cdot477}{k} = \frac{0\cdot55}{7\cdot2} \times \frac{1}{h}$$

$$\frac{1}{k} = \frac{0\cdot160}{h}$$

$$\frac{1}{h_1} = \frac{1\cdot013}{h}$$

The correction is only about one per cent.

The hypothesis that the stars are at the same distance is very far from true, and in consequence the corrections here determined are only rough; but the calculation is sufficient to show that when the motions are small and not very well determined the effect of the accidental errors may be quite appreciable.

Of the possible systematic errors, the most important are those due to an error in the adopted constant of precession and an error in the adopted motion of the equinox. The former would lead to an apparent rotation of the stellar system about the pole of the ecliptic; the

latter to a rotation about the pole of the equator. There is no way of determining the constant of precession except by a discussion of stellar motions; the motion of the equinox can, however, be determined from a discussion of observations of the Sun, and it is a matter for consideration how much weight ought to be attached to the solar and stellar determinations, respectively. The recognised determinations of these two constants have been based on the principle of haphazard velocities; on the two-stream theory the solution would become extremely difficult, and to some extent indeterminate, if the velocities and proportions of mixture of the two streams are not exactly the same all over the sky. The same difficulty occurs in defining the constant of precession as in defining the solar motion; though in this case the difficulty is a practical and not a philosophical one.* In practice it is, within reasonable limits, arbitrary how much of the observed motions shall be attributed to the rotation of the axes of reference, and how much to the heavenly bodies themselves. The only guide is that the residual stellar motions should follow as simple a law as possible. But, when it is certain that no really simple law is possible, this is not a condition that can be expressed and used analytically. When once the hypothesis of haphazard motions is given up, the constant of precession can only be given an approximate value, and there is no very satisfactory way of improving it.†

An investigation by Hough and Halm [6] throws important light on the relation between star-streaming and the precession-constant. It is shown by them that the inequality of mixture of the two drifts in different parts of the sky

* In dynamics we know precisely what we mean by absolute rotation though we may not be skilful enough to detect it; absolute translation cannot even be defined.

† A determination from the stars of the Orion type, which are supposed to be moving at random, would be of interest, but the inequality of distribution and the prevalence of moving clusters would make it difficult.

accounts for certain discordances in former investigations of the precession. But this work does not appear to lead to any means of determining the constant *de novo*.

Having regard to the practical impossibility of obtaining an accurate value of the precession-constant, there is an important advantage in proceeding so as to avoid the systematic errors arising therefrom. This is done when we treat two antipodal areas of the sky together; for the error of precession will be in opposite directions (in space) in the two areas, and its effect will be wholly or partially eliminated in the mean result.

The Maxwellian Law.

The Maxwellian or error-law plays an important part in the analysis both of the two-drift and ellipsoidal theory. The radial velocity determinations now available enable us to test in a direct manner how far the stellar motions obey this particular law.

For Type A the following table (21) compares the actual distribution of the radial motions (corrected for the solar motion) with an error-law.[7] In order to get rid of most of the effect of star-streaming, stars in the neighbourhood of the vertices, forty in number, were not used.

Table 21

Radial Motions of Type A.

Limits of Velocity.	Number of Stars.	
	Error Law.	Observed.
km. per sec.		
0·0 — 4·95	53·4	55
4·95— 9·95	46·2	47
9·95—15·95	38·3	30
15·95—25·5	27·4	30
25·5 —40	6·7	10
>40	0·2	0

A similar table (22) is given for Types II. and III. (F5—M). The distribution of observed velocities has been taken from a table given by Campbell.[8] No allowance is made for the effect of star-streaming, but that will not have so much influence proportionately as on the smaller motions of Type A.

TABLE 22.

Radial Motions of Types F5—M.

Limits of Velocity.	Number of Stars.	
	Error Law.	Observed.
km. per sec.		
0— 5	135	162
5—10	127	131
10—15	114	124
15—20	97	102
20—25	78	52
25—30	59	39
30—35	42	33
35—40	29	17
40—50	30	31
50—60	10	11
60—70	2	7
70—80	1	4
>80	0	10

For Type A the agreement of the observed distribution of the velocities with the error-law is remarkably close. For Types F5—M the table shows that the correspondence is not so good. The observed distribution has what is technically called a *positive excess*, that is to say, there are too many small and too many large motions compared with the number of moderate motions. An increase or decrease of the modulus of the error distribution, with which it is compared, will make the agreement better at one end but worse at the other end of the table. A distribution of this kind would be obtained if we mixed together error distributions having

different moduli; and it may therefore be supposed that the deviations arise from the non-homogeneity of the material used. It is also probable that if precautions had been taken to avoid the effects of star-streaming, the excess of large motions would have been much less pronounced.

We may conclude that in a selection of stars really homogeneous as regards spectral type (and perhaps luminosity also) the components of motion perpendicular to the line of star-streaming are distributed according to the error-law, as required by the two-drift and ellipsoidal hypotheses. But, in a selection of stars made under ordinary practical conditions, there is likely to be an excess of very large and very small proper motions, and a defect of moderate motions.

References.—Chapter VII.

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5. Weersma, *Astrophysical Journal*, Vol. 34, p. 325.
6. Hough and Halm, *Monthly Notices*, Vol. 70, p. 584.
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8. Campbell, *Stellar Motions*, p. 198.

Bibliography.

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Kapteyn, *Monthly Notices*, Vol. 72, p. 743.

Eddington, *Monthly Notices*, Vol. 67, p. 34.

Schwarzschild, *Göttingen Nachrichten*, 1907, p. 614.

For the direct methods, avoiding trial and error (not recommended).

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CHAPTER VIII

PHENOMENA ASSOCIATED WITH SPECTRAL TYPE

If two stars, one of Type A and the other of Type M, are chosen at random out of the stars in space, it may be confidently predicted (1) that the Type A star will be the more luminous of the two, and (2) that it will have a smaller linear velocity than the Type M star. We say intentionally "out of the stars in space," because, for example, the stars visible to the naked eye are a very special selection by no means representative of the true distribution of the stars. The odds are considerable in favour of both predictions being correct, though a failure may sometimes occur. Similar illustrations with other kinds of spectra might be given. In short there is a conspicuous correlation, on the one hand, between spectral type and luminosity, and, on the other, between spectral type and speed of motion. The former relation is scarcely surprising, and some correlation would be expected on physical grounds, although, perhaps, not so close as that actually found; but the connection between type and speed is a most remarkable result.

The discovery of the latter relation has come about very gradually. So early as 1892, W. H. Monck[1] pointed out that the stars of Type II. had larger proper motions on the average than those of Type I. Further research, especially by J. C. Kapteyn,[2] emphasised the importance of this discovery. It is well shown by the stars having

excessive proper motions. In a list given by Dyson[3] of ninety-five stars with annual proper motions of more than 1″, there are fifty-one of which the type of spectrum is known; of these, fifty are of Type II., and only one (Sirius) is of Type I. Again, of those with proper motions exceeding 0″·5, 140 belong to Type II. and four to Type I. It was realised that this phenomenon did not necessarily signify a connection between spectral type and the true linear speed; and there was a general preference for the less startling explanation that it was due to the feeble luminosity and consequent nearness of stars of the second type. Certain investigations of the parallactic and cross proper motions, as well as of the radial velocities, appeared to confirm this view.

The next stage was reached in 1903, when E. B. Frost and W. S. Adams[4] published their determinations of the radial velocities of twenty stars of the Orion type; it was shown that these stars have remarkably small linear velocities, averaging (for one component) only seven kilometres per second. This result seems to have been regarded as showing that the Orion stars were exceptional; apparently it was not suspected that this was a particular case of a general law.

With the introduction of the two-stream hypothesis, and consequent methods of investigation, fresh light was thrown on the subject. It was found that the "spread" of the motions of the Type I. stars was less wide than those of Type II., the former following much more closely the directions of the star-streams.[5] Though other interpretations were conceivable, this seemed to indicate that the individual motions of Type I. were smaller than those of Type II. Definite evidence was at length forthcoming in 1910 from the results of determinations of the radial velocities, which clearly showed that the speeds of the second type stars were larger on the average. But the radial velocity results led to a wider generalisation.

J. C. Kapteyn[6] and W. W. Campbell[7] pointed out independently that the average linear velocity increases continually as we pass through the whole series from the earliest to the latest types, *i.e.*, in the order B, A, F, G, K, M. The following table contains the results of Campbell's discussion.

TABLE 23.

Mean Velocities of Stars (Campbell).

Type of Spectrum.	Radial Velocity.	Weight (No. of Stars).
	km. per sec.	
B	6·52	225
A	10·95	177
F	14·37	185
G	14·97	128
K	16·8	382
M	17·1	73
Planetary nebulæ	25·3	12

The velocities for F, G, and K come in the right order, but it would be straining the figures too far to attach much importance to this. The rise from B to A and from A to Type II. (F, G, K), is quite well marked, and a rise from Type II. to M is fairly indicated. The position of the planetary nebulæ at the end is distinctly curious. If we have entire confidence in the law that the speed increases with the stage of development, it follows that a planetary nebula must be regarded as a final stage—certainly not as the origin of a star. There is some justice in a remark of R. T. A. Innes[8]: "The fact that we have seen a star change into a nebula* ought to outweigh every contrary speculation that stars originate from nebulæ." It is necessary to proceed cautiously in such an application; but we seem to have within our grasp a new method of deciding doubtful questions as to the order of development of the different stages in a star's history.

* Referring to the phenomena of the later stages of a Nova.

The residual motions given in Table 23 are corrected for the solar motion, but not for the star-stream motions. They do not therefore represent what we consider to be the actual individual stellar motions, as distinguished from the systematic motions. To remove the latter must affect the numbers appreciably. If, following Schwarzschild's hypothesis, a is the mean speed at right angles to the star-stream direction, and c the mean speed towards or away from the vertex, the mean radial speed at a point distant θ from the vertex is

$$\sqrt{a^2 \sin^2\theta + c^2 \cos^2\theta},$$

and the mean radial speed over the whole sphere is

$$\frac{1}{4\pi}\int\int \sqrt{a^2 \sin^2\theta + c^2 \cos^2\theta} \sin\theta\, d\theta\, d\phi.$$

This would be slightly modified by the fact that more stars are observed near the galactic plane than in other parts of the sky, but, since the axis of preferential motion lies in the galactic plane, the effect of this inequality is minimised.

Performing the integration, the mean radial speed becomes

$$\frac{1}{2}\left\{c + a\frac{\sinh^{-1}\beta}{\beta}\right\}$$

where

$$\beta = \sqrt{\frac{c^2 - a^2}{a^2}}.$$

If, for example, $\frac{a}{c} = 0{\cdot}56$, which is probably about true for the Type A stars, this speed is equal to $1{\cdot}30a$. So that to obtain the true *motus peculiaris* free from the effects of star-streaming we should divide the result for Type A given in Table 23 by 1·30. For the later types the velocity ellipsoid is less prolate, and the divisor would be smaller, about 1·15 ; Type B shows no evidence of star-streaming and the velocity already given may

remain unaltered. The mean individual speeds thus modified would then run—

B, 6·5 ; A, 8·4 ; F, G, and K, 13·6 kilometres per second.

There is still a steady increase of speed with advancing type, though the main jump is now between A and F.

Similar results have been obtained by Lewis Boss [9] from a discussion of the proper motions of stars. The method, which had previously been applied by Kapteyn to the Bradley proper motions, depends on the following principles. Let the proper motion be resolved into two components, the parallactic motion ν towards the solar antapex and the cross proper motion τ at right angles to it. We can determine the mean parallax of the stars of any type from the mean parallactic motion, by the aid of the known speed of the solar motion. By means of this mean parallax, the mean value of τ, regardless of sign, can be converted into linear measure. These linear cross-motions are exactly comparable with the radial motions that have just been discussed. Like them they are free from the effects of solar motion, but are not corrected for the star-streaming. Boss's results, which depend on the excellent data of his catalogue, are as follows :—

TABLE 24.

Mean Velocities of Stars (Boss).

Type.*	Cross Linear Motion.	Weight (No. of Stars).
	km. per sec.	
B	6·3	490
A	10·2	1647
F	16·2	656
G	18·6	444
K	15·1	1227
M	17·1	222

* In Boss's classification, B includes Oe 5 to B 5 ; A includes B 8 to A 4 ; F includes A 5 to F 9.

The close agreement with the quite independent evidence of the radial velocities is very satisfactory. Boss's results depend on the assumption that the solar motion is the same for all types, which is open to some doubt. As regards the irregularity of the progression F, G, K, there is little doubt that his method of excluding stars of excessive proper motion leads to too small a value of the parallactic motion as compared with the cross motion; and this is especially the case for Types F and G, which contain by far the largest proportion of great proper motions. The linear motions deduced by him for these two types should accordingly be diminished.

The facts here brought before us direct attention to the very deep-lying question,—How do the individual motions of the stars arise? It appears that as the life-history of a star is traced backwards, its velocity is found to be smaller and smaller. In the Orion stage it is only a third of what it will ultimately become. Must we infer that a star is born without motion and gradually acquires one? I believe this is the right conclusion, although there is more than one loophole of escape, which deserves consideration.

J. Halm[10] has suggested that equipartition of energy holds in the stellar system; according to his view the Orion stars move slowly, not because they are young, but because they are massive. If the stars were all formed about the same epoch, the large stars might be expected to take longer to pass through their stages of development than the smaller stars, so that at the present time the more massive the star the earlier would be its spectral type. The main direct evidence as to the masses of the stars is found in a discussion of the spectroscopic binaries of which the orbits have been investigated. In cases where both components are bright enough to show their spectra the quantity $(m_1 + m_2) \sin^3 i$ can be found; here i is the unknown inclination of the orbit to the plane of the sky.

There are available for discussion seven binaries of Type B and nine of Types A—G. Assuming that the mean value of $\sin^3 i$ will be the same for both groups, it is found that

$$\frac{\text{average mass of Type B binaries}}{\text{average mass of other types}} = 8{\cdot}6.$$

When the spectrum of one component only can be observed the quantity

$$m_1\left(\frac{m_1}{m_1 + m_2}\right)^2 \sin^3 i$$

can be found. There are seventy-three suitable orbits of this kind known. These give

$$\frac{\text{average mass of Type B}}{\text{average mass of other types}} = 6{\cdot}5.$$

These results indicate that the B stars are considerably more massive than the other types; and the ratio actually agrees with that demanded by the law of equipartition of energy, viz., the average mass is inversely proportional to the square of the average velocity. But the main argument for equipartition has been a theoretical one, depending on a supposed analogy between the behaviour of stars and the molecules of a gas. This subject will be considered in Chapter XII.; the evidence there given seems convincing that the analogy of the stellar system with a gas system does not hold good; and equipartition, if it exists, cannot be explained in this way.

It seems certain that the motion of a star has not during the period of its existence been appreciably disturbed by the chance passage of neighbouring stars. This doctrine of *non-interference* leads to the conception that each star describes a smooth orbit (not necessarily closed) under the central attraction of the whole stellar system. Such a star will wander sometimes near the

centre, sometimes at a remote distance, transforming potential into kinetic energy and *vice versâ*. The nearer it is to the centre the greater will be its speed. Consequently in the stellar system the average speed may be expected to diminish from the centre outwards. This conclusion depends on the view that most of the stars are continually approaching and receding from the centre; if the majority were describing circular orbits the speed would actually increase from the centre outwards. But accepting it as a possible and fairly likely condition, it offers another explanation of the association between velocity and spectral type. Suppose the Orion stars move slowly, not because they are young, but because they are very distant. The order of spectral type is (or was until recently) believed to be the order of luminosity and, consequently, for stars down to a limiting magnitude, the order of mean distance. Thus it may be that we are using the spectral classification as a distance classification, and determining a relation between distance and speed.

This explanation was formerly put forward in a tentative manner by the writer,[11] but it is given here only that it may be disproved. To test it, the radial motions of the stars of Type A were taken and grouped according to the magnitude of the proper motion. This grouping is a rough division according to distance, since the larger proper motions usually indicate the nearer stars.

Centennial Proper Motion.	Mean Radial Velocity.	No. of Stars.
″	km. per sec.	
>20	10·1	19
12—20	8·8	29
8—12	12·4	38
4—8	11·6	61
0—4	11·1	65

There is here no sign of a decreasing speed with increasing distance. It is clear that distance cannot be the determining factor.[12]

We are thus thrown back on the original and straightforward conclusion that the phenomenon is a genuine correlation between speed and spectral type, independent of either mass or distance.

We have up to now been discussing the relation between spectral type and the individual stellar motions; it remains to be considered whether the systematic motions vary from one type to another. It was found in Chapter V. that the declination of the solar apex depended on the type of stars chosen, being more northerly for the later types. It is not clear whether the speed of the solar motion is appreciably different. The following results are given by Campbell,[13] but the amount of data is scarcely sufficient to allow of much weight being attached to them :—

Type.	Solar Velocity.	No. of Stars.
	km. per sec.	
B	20·2	225
A	15·3	212
F	15·8	185
G	16·0	128
K	21·2	382
M	22·6	73

According to the two-drift hypothesis the solar or parallactic motion is merely the mean of two partially opposing drift-motions, and for a fuller understanding of these changes, or possible changes, of the solar motion reference must be made to the drifts. It has been found by many independent researches that the star-streaming tendency is scarcely shown in the Type B stars, that

it is most strongly shown in Type A, and it becomes less marked in succeeding types, though still quite prominent in Type K. The sudden development of the star-streaming in its full intensity in passing from Type B to Type A is a curious phenomenon, but the evidence for it is overwhelming.

The question has been studied quantitatively by H. A. Weersma[14] from the data of Boss's Catalogue. If V is the velocity of one drift relative to the other, Ω the mean individual speed of the stars, he finds,

For Type A	$\frac{V_1}{\Omega_1}$	=	2·29	±	0·19
For Types K and M .	$\frac{V_2}{\Omega_2}$	=	0·98	±	0·11

Again if P be the solar motion relative to the mean of the stars,

For Type A	$\frac{P_1}{\Omega_1}$	=	1·08	±	0·08
For Types K and M .	$\frac{P_2}{\Omega_2}$	=	0·62	±	0·04

It was assumed in the investigation that the proportion in which the stars are divided between the two drifts is the same for Type A as for K and M, viz., 3 : 2. It is by no means certain that this is correct.

These differences between the quantities V/Ω, P/Ω for the two groups are largely accounted for by the differences in Ω that have already been discussed; but it would appear that to reconcile the results we must have also P_1 different from P_2 or else V_1 different from V_2. Having regard to the probable errors the evidence for this is rather slight. If, for example, we put $\Omega_2 : \Omega_1 = 1{\cdot}8$, a value which appears to represent the results derived from the radial velocities (p. 158), then

$$V_1 : V_2 = 1{\cdot}30$$
$$P_1 : P_2 = 0{\cdot}97$$

which makes the solar motion about equal for the two

types, and gives a real diminution of the star-stream velocity, in passing from Type A onwards. With a somewhat smaller ratio $\Omega_2 : \Omega_1$ we should obtain the same star-stream velocity, but a smaller solar motion for Type A than Types K and M,—an equally likely explanation, which, moreover, receives a little support from the direct determinations of the solar motion already quoted.

The view favoured by Kapteyn[15] abandons the assumption that the division between the two drifts is the same for all types. Instead there is a continuous increase in the proportion of Drift II. stars as the spectral type advances, and at the same time a continuous change in the direction of the stream motions. He considers that in the course of time the stream-motions have slightly changed in such a way that the oldest stars have deviated most and the youngest least, but all in a higher or lower degree, from the original direction and velocity.

The conspicuous relation between the luminosities of the stars and their spectral types has already been touched upon in discussing the nearest stars. Much further information can be gained from investigations of the general mass of the stars. In consequence of the fact that we usually consider catalogues or selections of stars limited by a certain apparent magnitude, the difference in luminosity leads to a difference in the average distance of the spectral classes. In saying, as we commonly do, that the B stars are more remote than the A stars, we do not mean that there is any difference in their real distribution in space, but only that, when we consider stars limited by a certain magnitude, the selection of B stars is dispersed through a larger volume of space than the selection of A stars.

We might hope to gain information as to the average distances, and therefore as to luminosities of the spectral types, by comparing their degrees of concentration to the

galactic plane. The general tendency of the stars to crowd to the galactic plane is explained by the oblate shape of the stellar system, so that we see through a greater depth in some directions than in others. But, clearly, if a class of stars is confined to a small sphere in the centre of the stellar system, its distribution will not in any way be affected by the shape of the boundary. Thus we find that the stars with proper motions greater than 10″ per century show no galactic concentration ; they are all comparatively near to us. The greater the average distance of the stars, or the wider the volume of space through which they can be seen, the more will the oblate shape of the system affect them. The deficiency of stars in the region of the galactic poles will be more and more marked. Thus we may expect the amount of galactic concentration to be a measure of the average distance of the class.

From the Revised Harvard Photometry, E. C. Pickering [16] has determined the distribution of the stars down to a limiting magnitude of 6·5, arranged according to spectral type and galactic latitude. His results are given in Table 25.

TABLE 25.

Distribution of the Stars brighter than $6^{m}\cdot5$.

Zone.	Mean Galactic Latitude.	B.	A.	F.	G.	K.	M.
	°						
I.	+62·3	8	189	79	61	176	56
II.	+41·3	28	184	58	69	174	49
III.	+21·0	69	263	83	70	212	57
IV.	+ 9·2	206	323	96	99	266	77
V.	− 7·0	161	382	116	84	239	45
VI.	−22·2	158	276	117	100	247	69
VII.	−38·2	57	161	94	59	203	59
VIII.	−62·3	29	107	77	67	202	45

The eight zones are of equal area, so that the numbers show directly the relative density at different galactic latitudes.

Pickering's division of the spectral types was as follows: B = O – B8; A = B9 – A3; F = A4 – F2; G = F5 – G; K = G5 – K2; M = K5 – N. The divisions are somewhat different from those we have previously considered.

Taking the degree of concentration shown in these tables as a measure of average distance, we should arrange the types in the order of decreasing distance and decreasing luminosity, thus:

$$\text{B, A, } \overbrace{\text{F and G}}, \text{ K, M,}$$

which is identical with the order of evolution usually accepted.

This agrees well with the results of Chapter III. as to the luminosity of the stars derived from parallax investigations. A general decrease in luminosity with advancing type was there noted. Further, as the sequence B, A, F, G, K, M is probably the order of decreasing temperature, it is not surprising that the luminosity should decrease in the same way.

Nevertheless, this order is undoubtedly wrong. It is not difficult to measure the average distances of stars of the spectral types by less hypothetical methods. The mean parallactic motion in arc is proportional to the mean parallax, for the true linear parallactic motion is, at least approximately, the same for all the spectral classes. Or, again, by comparing the mean cross-proper motion (at right angles to the parallactic motion) in arc with the mean cross-motion in linear measure (Table 23), an independent determination of the mean distance is obtained. Five investigations on these lines may be cited.[17]

TABLE 26.

Mean Distances of the Spectral Types.

(*a*) L. Boss.

Type.	Parallactic Motion.	No. of Stars.
	″	
Oe5—B5	2·73	490
B8—A4	4·08	1647
A5—F9	4·99	656
G	3·12	444
K	4·03	1227
M	3·29	222

(*b*) J. C. Kapteyn.

Type.	Mean Parallax.	No. of Stars.
	″	
B	0·0068	440
A	0·0098	1088
F, G, K	0·0224	1036
M	0·0111	101

(*c*) W. W. Campbell.

Type.	Mean Parallax.	No. of Stars.
	″	
B0—B5	0·0061	312
B8, B9	0·0129	90
A	0 0166	172
F	0·0354	180
G	0·0223	118
K	0·0146	346
M	0·0106	71

(*d*) H. S. Jones.

Type.	Mean Parallax.	No. of Stars.
	″	
B0—B5	0·0031	11
B8—A4	0·0058	188
A5—F9	0·0110	187
G0—G5	0·0076	141
G6—M	0·0056	140

(*e*) K. Schwarzschild.

Approximate Type.	Colour Index.	Parallactic Motion.	No. of Stars.
	m.	″	
B	−0·65	3·5	64
A	−0·35	2·9	332
F	−0·05	8·9	277
G	+0·25	20·8	150
	+0·55	8·6	126
K	+0·85	7·6	277
	+1·15	4·9	199
	+1·45	4·0	184
M	+1·75	4·6	71

The parallactic motion (centennial) is 410 times the parallax.

(*a*) L. Boss's results, based on the proper motions of stars brighter than $6^m \cdot 0$ in his Catalogue, refer to much the same stars as those used in Pickering's discussion of galactic distribution. Unfortunately Boss rejected all proper motions greater than $20''$ per century; this has not only made his values systematically too small, but it has had a disproportionately great effect in the case of Types F and G, which include the bulk of the stars with excessively great motions. Accordingly the values for F and G need to be greatly increased.

(*b*) Kapteyn's results depend on less accurate proper motions than the preceding. To allow for differences in the mean magnitudes of the different types, the values of the mean parallax have been corrected so as to correspond to magnitude 5·0.

(*c*) Campbell's determination is based on the cross-motions. It refers to stars somewhat brighter than the other investigations, the mean magnitude being $4^m \cdot 3$.

(*d*) Jones's determination depends on the parallactic motions of stars between Dec. $+73^\circ$ and $+90^\circ$, of mean magnitude $6^m \cdot 8$. The difference of 2·5 magnitudes between these stars and Campbell's accounts for the smaller parallaxes found.

(*e*) Schwarzschild's classification is primarily by colour-index. The proper motions were taken from Boss's Catalogue.

All the investigations agree in showing that the mean parallax increases steadily from Type B to a point somewhere about F or G, and then decreases again to a small value for Type M. The order of distance is thus altogether different from the standard order B, A, F, G, K, M. In particular it appears that the stars of Type M are more distant than any other type except B.

How then is it that the M stars show practically no galactic concentration, whereas the A stars are strongly condensed? Our previous explanation fails, because the

assumption that Type M is much less remote than Type A is now shown to be false. It seems necessary to conclude that the apparent differences in galactic distribution are real; that the system of the A stars is very oblate, and the system of Type M is almost globular.

This leads to the following theory. The stars are formed mainly in the galactic plane. Type B, on account of the low individual speeds and the short time elapsed since birth, remains strongly condensed in the plane. In succeeding stages the stars have had time to stray farther from the galactic plane, and their higher velocities assist in dispersing them from it. In the latest type, M, the stars have become almost uniformly scattered, and very little trace remains of their original plane. We shall see reason in Chapter XII. to modify this hypothesis slightly.

It will be seen that, in regard to the relation of spectral type both to speed and to galactic concentration, we have been driven to adopt the straightforward interpretation of the phenomena that occurs most naturally to anyone who has not considered the subject deeply. The correlation is exactly what it appears to be, and subtle suggestions as to its being mixed up with other effects are found to fail in the end. Yet I think we have been right in not jumping to the obvious conclusion at once; it was necessary to examine, and for a time prefer, the alternative explanations, which, though more complex in themselves, led to a simpler conception—too simple it now appears—of the stellar system.

An outstanding point of great difficulty remains, which may most conveniently be illustrated by the stars of Type M. We have been led to two opposed views as to their luminosity. In the parallax investigations of Chapter III. they were found to be the faintest of all the types; in the present statistical investigations they are found to be the most luminous, except Type B. Our

conclusion from the parallax investigations may be judged to rest on rather slight though very consistent evidence; but other (less trustworthy) parallaxes confirm it, and moreover the K stars show a similar discordance in the two kinds of investigation. It may be admitted at once that the parallax and statistical results relate to entirely different selections of stars; none of the extremely feeble M and K stars of Tables 3 and 5 enter into the data of our last discussion. Both results are probably right; but it is difficult to see how they are to be reconciled.

The leading contribution to this problem is the hypothesis of "giant" and "dwarf" stars put forward by E. Hertzsprung [18] and H. N. Russell.[19] They consider each spectral type to have two divisions, which are not in reality closely related. The one class consists of intensely luminous stars and the other of feeble stars, with little or no transition between the two classes. Assuming that the feeble stars are much more abundant than the luminous stars in any volume of space, the parallax investigations will take hold of the dwarfs mainly, whilst the statistical investigations, selecting by magnitude, will be concerned with the giants. This will account for the different luminosities; for Type M will then denote two entirely different classes in the two kinds of research.

Russell has supported this hypothesis by direct evidence from the parallax determinations. By his kindness I am permitted to reproduce his diagram (Fig. 19) of the absolute luminosities of all stars for which the necessary data could be obtained; parallaxes have been used to which we should hesitate to attach much weight; but the principal features of his diagram can scarcely be doubted. Below each spectral type are shown dots which represent on a vertical scale the absolute magnitudes (magnitude at a distance of 10 parsecs) of individual stars of that type. The large circles represent mean values for bright stars of small proper motion and parallax.

The general configuration of the dots seems to correspond to two lines thus, ⋎. There would appear to be two

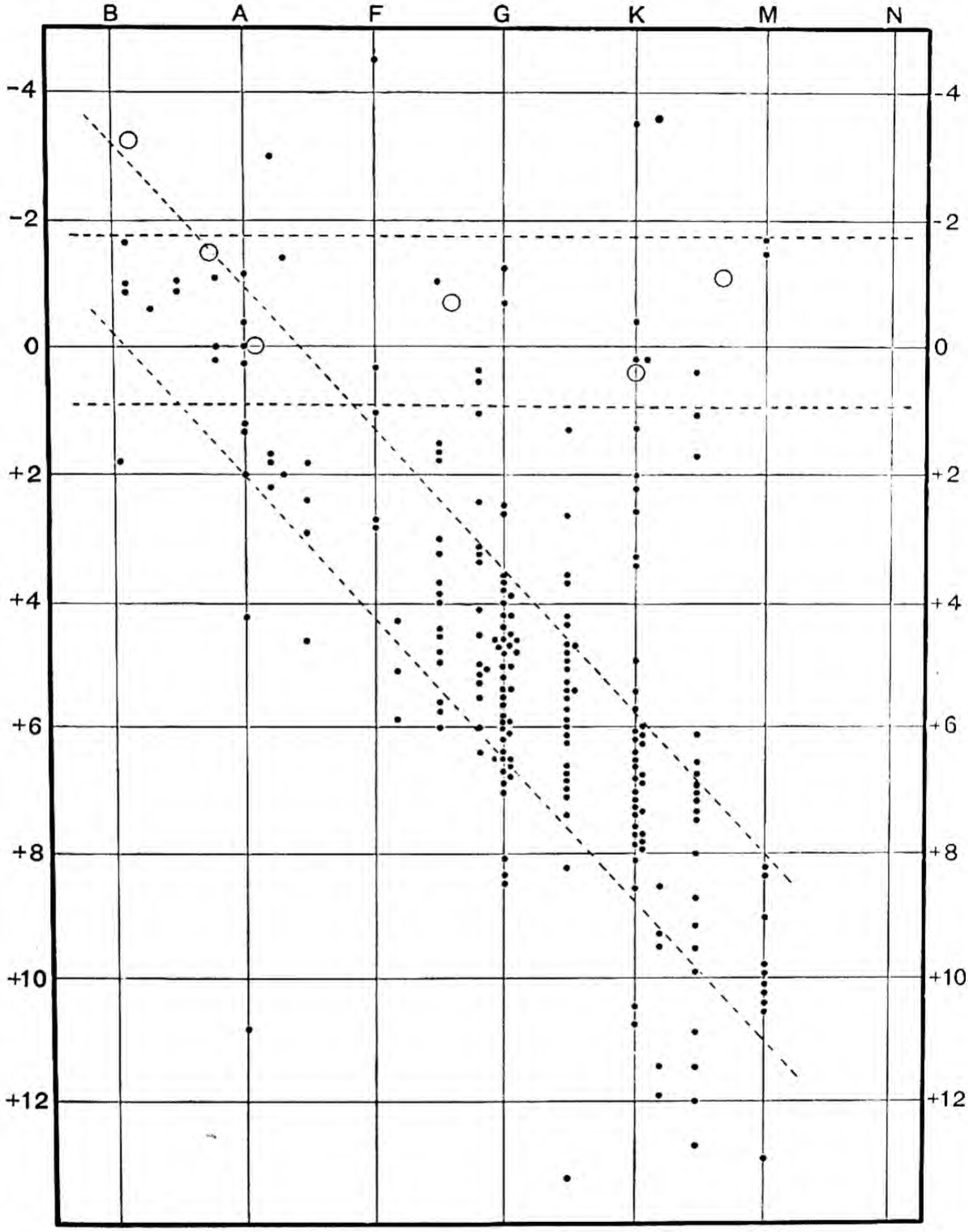

Fig. 19.—Absolute Magnitudes of Stars (Russell).

series of stars, one very bright and of brightness almost independent of the spectrum, and the other diminishing

rapidly in brightness with increasing redness. The former series, corresponding to the horizontal line, are the giants, and the latter, corresponding to the oblique line, the dwarfs. For Types B and A the giants and dwarfs practically coalesce; but the divergence increases to a very large amount at Type M. It must be remarked that the evidence of this diagram, convincing as it looks, does not compel us to divide Types K and M into two distinct classes. The stars of which the parallax and luminosity have been measured are in most cases chosen for brightness or for nearness (large proper motion). The two groups may thus result from the double mode of selection, without implying any real division in the intrinsic luminosities.

For example, if the absolute magnitudes M are distributed according to the frequency law

$$e^{-k^2(M-M_0)^2},$$

the stars of absolute magnitude M and of apparent magnitude greater than m are those within a sphere of radius r given by

$$\log_{10} r = 0{\cdot}2\,(m-M),$$

the volume of this sphere is proportional to r^3 or to

$$10^{0{\cdot}6\,(m-M)},$$

and the frequency of an absolute magnitude M among stars limited by the magnitude m is proportional to

$$e^{-k^2(M-M_0)^2+1{\cdot}38\,(m-M)}.$$

This is an error distribution with the same dispersion as before but ranged about the mean value $M_0 - \dfrac{0{\cdot}69}{k^2}$.

Thus our two methods of selecting parallax stars would give luminosities clustering about two separate magnitudes M_0 and $M_0 - \dfrac{0{\cdot}69}{k^2}$. If it is supposed that $1/k$ increases with advancing type, the two diverging groups will be explained. From Fig. 19 it appears that the two groups

of Type M differ by about eleven magnitudes. Setting $0{\cdot}69/k^2 = 11$, we have $1/k = 4^m{\cdot}0$. For Type G the difference is six magnitudes, and $1/k = 2^m{\cdot}9$.

Russell has shown that the error-law, assumed for the absolute magnitudes, is confirmed by the observations; but the modulus is smaller than that calculated, viz., $1/k = 1^m{\cdot}6$ (corresponding to a probable deviation $0^m{\cdot}75$). This result may be considered to refer to the mean of all spectral types.

Whilst the evidence from the directly determined luminosities is thus scarcely conclusive, there are several other indications which point to a real existence of the two series. Perhaps the strongest argument is a theoretical one. According to the well-known theories of Lane and Ritter, as a star contracts from a highly diffused state, its temperature rises until a certain concentration is reached, after which the loss of heat by radiation is greater than the gain by conversion of gravitational energy into heat, and the star begins to cool again. Recent discoveries of a new supply of energy in radio-active processes involve some modification of these theories; but probably the general result of a temperature increasing to a maximum, and then diminishing, may be accepted. Now, if the spectrum of a star depends chiefly on its effective temperature, it may well be that the Draper classification groups together stars of the same temperature regardless of whether they are in the ascending or descending state. The former would be highly diffused bodies, and the latter concentrated. The surface brightness, which depends on the temperature, being the same, the ascending stars with large superficial area would give a great deal more total light than the denser descending stars. If then the stars have all much the same mass—a conclusion supported by such evidence as is available—we shall have two groups in each type, one of low density and intense total luminosity and one of high density and

low luminosity. These two groups will coalesce for the B stars, which mark the maximum temperature reached, and fall wider apart the lower the temperature, just as in the diagram. Moreover, on the ascending side the increasing temperature and diminishing superficial area will oppose one another in their effect on the luminosity, so that the change of brightness from type to type will be small. On the descending side the decreasing surface and decreasing surface-brightness will both lead to a rapid change of luminosity from type to type.

The determinations of density of the visual and spectroscopic binaries (p. 24) favour the view that some of the later type stars are in a very diffused condition, and that others are very condensed. Table 27, due to H. Shapley,[20] contains the determinations of density of eclipsing systems. Unfortunately, these are mainly early type stars, but the tendency to divide into two groups is well illustrated even in F and G.

TABLE 27.

Stellar Densities (Shapley).

Density (Water = 1).	B.	A.	F.	G.	K.
>1·00	—	—	—	1	—
1·00 —0·50	—	—	1	—	—
0·50 —0·20	1	10	6	1	—
0·20 —0·10	4	12	1	1	—
0·10 —0·05	3	17	—	—	—
0·05 —0 02	2	8	—	—	—
0·02 —0·01	—	3	1	1	—
0·01 —0·001	2	—	—	—	—
0·001—0·0001	—	—	—	2	1
<0·0001	—	—	1	1	—

It is well-known that Lockyer's classification differentiates the stars into series of ascending and descending temperatures, but according to Russell the giant and dwarf stars do not correspond to Lockyer's criteria.

With regard to the possibility of distinguishing two series by slight differences in their spectra an interesting contribution has been made by Hertzsprung. In Miss Maury's classification certain stars are discriminated as having what is known as the *c* character, marked by the very sharply defined appearance of the absorption lines. These stars (which are rather few in number) are found to have much smaller proper motions than the corresponding stars of the same spectral type without the *c* characteristic. With one exception (υ Ursae Majoris) the motions are almost imperceptible, generally smaller even than those of the early Orion type. If, in order to allow for differing magnitudes, the proper motions are all multiplied so as to represent what would be the apparent motion of the star at a distance at which it would appear of zero magnitude, the following (condensed from Hertzsprung's table [21]) shows the results for stars with and without the *c* characteristic :—

TABLE 28.

Stars with the c *Characteristic.*

c-Star.	Proper Motion.	Normal P.M.	*c*-Star.	Proper Motion.	Normal P.M.
	″	″		″	″
o_2 Can. Maj. . .	0·03	0·20	ν Persei	0·06	1·20
67 Ophiu. . . .	0·08	0·20	α Persei	0·09	2·22
Rigel	0·00	0·41	δ Can. Maj. . .	0·02	3·25
μ Sagittarii . .	0·03	0·41	ρ Cassiop. . . .	0·07	3·25
2 Camelop. .	0·05	0·52	γ Cygni	0·01	3·25
η Leonis . . .	0·03	0·39	Polaris	0·11	3·25
α Cygni	0·01	0·44	η Aquilæ . . .	0·07	0·48
22 Androm. . .	0·09	0·72	α Aquarii . . .	0·07	0·48
α Leporis . . .	0·02	0·57	10 Camelop. . .	0·08	0·48
π Sagittarii . .	0·14	0·57	δ Cephei . .	0·08	0·48
υ Ursæ Maj. . .	1·96	0·57	ζ Geminorum .	0·02	0·48
ϵ Aurigæ . . .	0·07	1·20			

(The proper motions are reduced to zero magnitude as standard.)

The second column gives the proper motion of the *c*-star, the third column the average proper motion of the

remaining stars of the same type in Miss Maury's classification, in each case reduced to zero magnitude. The first five stars are of Type B0 to B9, the next six are from A to F and the remainder are in sub-divisions of types F and G. There are no K or M stars in the table.

It is clear that these stars with *c*-characteristic must be very remote and therefore (with the exception of *υ* Ursae Maj.) belong to the class of "giants." It looks as though a beginning has been made in the direct discrimination of the two groups by their spectra.

It will be seen that a very strong case has been made out for the recognition of two divisions in the later spectral types, corresponding to widely different luminosities. Yet there is a great difficulty in accepting Russell's theory in its entirety; if it is right, it causes a revolution in many of the results that have been generally accepted. In particular we shall have to revise the supposed order of evolution which has hitherto been assumed. Russell's theory gives the complete order M_1 K_1 G_1 F_1 A_1 B A_2 F_2 G_2 K_2 M_2, where the suffix 1 refers to the giants and 2 to the dwarfs. It is an essential part of his theory that the dwarf stars of Types M and K are too faint to appear in the statistical investigations of proper motions and radial velocities; that is to say, types M and K must be identified with M_1 and K_1, so far as these investigations are concerned. Moreover, the values of the parallactic motion show that the dwarfs of Types F and G play a predominant part in such researches, for the average distance and intrinsic brightness of these types is much less than for Types B and A. Therefore, somewhere about Type G we cross over, as it were, from the ascending branch to the descending branch. Having regard to this, the order of evolution becomes M K B A F G; an order which applies to all researches depending on stars selected for magnitude. From the astrophysical point of view the apparent breach of continuity between K and B does not matter; it is not

suggested that stars really pass from K to B at a jump, but that the intermediate types are outnumbered in the catalogues by stars indistinguishable from them in spectra but in a later stage of evolution.

The new order M K B A F G upsets altogether the regular progression of speed with type, and of galactic concentration with type. We should have to suppose that a star is born with a large velocity, that the speed decreases almost to rest and then increases again. Even if we do not insist on the predominance of dwarfs in F and G (though by abandoning it we lose one of the advantages of Russell's theory), and are content with simply reversing the usual order of evolution, the difficulties are great. We should have to amend our former hypothesis, and suppose that the stars originate with large velocities in a nearly spherical distribution, and afterwards become concentrated to the galactic plane, losing their velocities as they do so. The explanation is not improved by being turned upside down. And, further, it was shown in Chapter III. that the feebly luminous stars—the dwarfs of Types K and M—have extremely large velocities, so that in still later stages the speeds must increase again, even beyond their original amounts. The fact that both dwarfs and giants of Types K and M have larger speeds than the other types seems to imply a close connection between them, and it is very unsatisfactory to place them at opposite ends of the evolutionary scheme.*

There is another piece of evidence which lends strong support to the generally accepted order of evolution. Table 29 shows the periods of the spectroscopic binaries arranged according to type; it is due to Campbell.[22]

* **For Russell's reply to these criticisms, see *The Observatory*, April, 1914, p. 165.**

TABLE 29.

Periods of Spectroscopic Binaries (Campbell).

Type.	Period.						Total.
	Short.	0^d—5^d	5^d—10^h	10^d–365^d	> 1 year	Long.	
O and B	8	15	10	14	1	0	48
A	4	10	1	12	2	0	29
F	0	6	2	4	3	1	16
G	0	0	0	1	6	3	10
K	0	0	0	2	3	9	14
M	0	0	0	0	1	1	2

The columns headed "short" and "long" contain stars the periods of which have not been determined.

The increase in period with advancing type is very striking. It is also significant that so large a proportion of the spectroscopic binaries are of early type, indicating that the later types generally move too slowly to be detected with the spectroscope. A rough classification by R. G. Aitken of the more rapidly moving visual double stars gave the following proportions :—

Types O and B	. . . .	4 stars.
,, A ,, F	. . . .	131 ,,
,, G ,, K	. . . .	28 ,,
,, M ,, N	. . . .	1 ,,

Thus components of Type B are rarely sufficiently far apart to be seen separated ; and it would appear that in Types M and N the separation is so great that they are almost unrepresented on Aitken's list. If we believe that double stars originate by fission, and that the components separate farther and farther under the influence of tidal and other forces,* as time advances, we cannot but regard this result as a thorough vindication of the standard order of the types. Moreover, in the case of the spectroscopic

* H. N. Russell has directed my attention to the fact that tidal forces are competent to produce only a limited amount of separation.

binaries at least, we are dealing with stars selected for brightness exactly as in the statistical investigations of stellar motions—just the selection for which the order is challenged by Russell's hypothesis.

To sum up the present position, there is direct evidence that in the later types the stars are of two grades of luminosity, and that they are of two grades of density. The former division might perhaps be due to two different principles of selection of the stars being employed, though that would leave unexplained why on the one principle the late type stars are the faintest, and on the other principle the brightest of the classes. If we associate the divisions of luminosity with the divisions of density, as the Lane-Ritter theory suggests, this upsets the usually accepted order of evolution, *so far as statistical investigations are concerned*, an order which has been independently confirmed by studies of stellar velocities, galactic distribution, and the periods of binary stars.*

We shall conclude this chapter with some general remarks on the Orion type and fourth type stars, both of which present some interesting features.

The position of the Orion or Type B stars is very remarkable. Neither from their proper motions nor from their radial velocities do they show any tendency to partake of the motions of the two star-streams. If we had to class them with one of the drifts by their motions, we should naturally assign them to Drift I., but that is only because the Drift I. motion approximates more nearly to the parallactic motion than Drift II. does. Actually, such systematic motion as there is seems to be purely parallactic and due to the motion of the Sun in space. We now know that this peculiarity of being at rest in space, except for small individual motions, is shared by other

*** Elsewhere in this book the point of view is always that of the older theory—not Russell's theory—unless expressly stated.**

stars not belonging to this type. As we have seen, the dissection of the stellar motions into two streams leaves over a certain excess of stars (including both A and K types) moving towards the solar antapex, and presumably therefore at rest when the parallactic motion is removed.

A notable feature of the Type B stars is their tendency to aggregate into moving clusters. "Moving Clusters" is perhaps rather a misnomer, for the motion is usually very small; but they are groups apparently analogous to the Hyades cluster. The great Scorpius-Centaurus cluster, the constellation Orion, the Pleiades and the Perseus cluster between them account for a considerable proportion of the known stars of this type. The distance of these groups appears to be from about seventy to one hundred parsecs, except perhaps in Orion, which may be more distant. The remaining stars of the type have generally considerably smaller proper motions than these cluster stars, and are judged to be more distant still. This was well shown in Fig. 4 in which the non-cluster stars, forming the group of crosses near the origin, have barely appreciable proper motions. Lewis Boss[23] finds from his discussion of proper motions that a space round the Sun having a radius of seventy parsecs (corresponding to a parallax $0''\cdot015$) is "almost wholly devoid of these stars." Such a space would, according to the conclusions of Chapter III., contain at least 70,000 stars of other types. There seems no reason to believe that the part of space around our Sun is unusually bare of Type B stars; it is rather to be supposed that their general distribution is extremely rare, but that owing to brightness they are visible at say ten times the distance of an ordinary star and therefore through a space a thousand times as great. Their distribution, though rare on the average, is irregular, and in the moving clusters there must be many comparatively crowded together.

The assumption that the proper motion gives a measure

of the distance of these stars is more than usually justifiable in this case. Owing to their small individual motions, and to the absence of star-streaming, the whole motion cannot generally differ much from the parallactic motion. Omitting the divisions B8 and B9 which are probably more closely allied with Type A, there is only one known case of a large proper motion, that of the star α Gruis with a motion of $20''\cdot2$. Its parallax (determined by Gill with great accuracy) is only $0''\cdot024$, so that its linear velocity must be remarkably great for its class. No other stars from B0 to B7 have centennial motions so great as $10''$. Of late Type B stars (B8 and B9) the motions of Regulus, $25''$, and of β Tauri, $18''$ per century, are unusually large.

The fourth type stars (Type N) are for the most part too faint to come into the general discussions of distribution and motions. In Pickering's table of the galactic distribution of the types, the few that were included were classed with the M stars. Happily there are too few of them to affect the figures appreciably, for they present a very strong contrast to Type M in their distribution. It has been shown by T. E. Espin [24] and J. A. Parkhurst [25] that they are strongly concentrated to the galactic plane, as the following table shows:—

TABLE 30.

Galactic Distribution of Type N.

Galactic Latitude.	Number of N-Stars.		Relative Density.		Density of Durchmusterung Stars.
	Espin.	Parkhurst.	Espin.	Parkhurst.	
° 0—5	123	92	11·4	18·3	2·7
5—10		46		9·2	2·6
10—20	43	58	4·0	6·0	2·1
20—30	27	17	3·0	1·9	1·5
>30	31	29	1·0	1·0	1·0

The concentration is even a little stronger than for the Orion stars, but allowance should be made for the fact that we are using a much fainter limit of magnitude for these than for Pickering's table. The fainter the magnitude and the greater the distance, the stronger should be the apparent concentration to the galactic plane, as is indeed borne out by observation. Making allowance for this, Type N may probably be placed between B and A in the order of galactic condensation.

From 120 stars of this type of average magnitude $8^m{\cdot}2$, J. C. Kapteyn [26] has determined the parallactic motion; this is found to be $0''{\cdot}30$ per century with a probable error of practically the same amount. The corresponding parallax would be $0''{\cdot}0007 \pm 0''{\cdot}0007$. For the Orion stars the parallax found by the same method was $0''{\cdot}0068 \pm 0''{\cdot}0004$ for magnitude 5·0 (agreeing with other determinations already quoted). These N stars are thus many times more remote than the Orion stars. Their luminosity, however, may be slightly less, the difference $3^m{\cdot}2$ in apparent magnitude counterbalancing the greater distance.

Hale, Ellerman, and Parkhurst [27] have pointed out that the fourth type stars possibly have certain features in common with the Wolf-Rayet type. But they saw no reason to believe that any important organic relationship connects the two types.

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24. Espin, *Astrophysical Journal*, Vol. 10, p. 169.

25. Parkhurst, *Yerkes Decennial Publications*, Vol. 2, p. 127.

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CHAPTER IX

COUNTS OF STARS

In the investigations described in the four preceding chapters we have generally been confined to stars brighter than the seventh magnitude. Occasional excursions have been made beyond that limit, and stars down to the ninth or tenth magnitudes have made some contribution to our knowledge; further than this we have been unable to go. Beyond the tenth magnitude there is an ever-increasing multitude of stars, which hold their secret securely. We know nothing of their parallaxes, nothing of their spectra, nothing of their motions. There is only one thing we can do—count them. Carefully compiled statistics of the number of stars down to definite limits of faintness can still yield information which is of value for our purpose.

The fundamental theorem relating to these statistics is as follows:

In a stellar system of unlimited extent in which the stars are scattered uniformly, the ratio of the number of stars of any magnitude to the number of stars one magnitude brighter is 3·98.

The ratio alluded to is usually called the *star-ratio.* If in any direction it is found that the star-ratio falls below the theoretical value 3·98, this shows that we have penetrated so far as to detect a thinning out in the density of distribution of the stars. It is assumed that the absorption of light in space is negligible.

The number 3·98 is equivalent to $(2{\cdot}512)^{\frac{3}{5}}$ and the formula may be written:

$$\text{star-ratio for one magnitude} = (\text{light-ratio for one magnitude})^{\frac{3}{5}}.$$

In this form the theorem becomes fairly evident. A light ratio of 2·512 involves a distance-ratio of $(2{\cdot}512)^{\frac{1}{2}}$, and a volume-ratio $(2{\cdot}512)^{\frac{3}{2}}$. That is, for every small volume of space S at a distance D, there will be a corresponding volume $(2{\cdot}512)^{\frac{3}{2}}S$ at a distance $(2{\cdot}512)^{\frac{1}{2}}D$, such that the distribution of apparent magnitudes of the stars in the two volumes will correspond except for a difference of one magnitude due to the distance factor. But there will be $(2{\cdot}512)^{\frac{3}{2}}$ times as many stars in the second volume as in the first. Hence, a drop of one magnitude multiplies the number of stars by the factor 3·98 through the whole range. The fact that the stars are of varying degrees of intrinsic brightness is taken into account in this argument.

The thinning-out of the stars at great distances from the Sun manifests itself in a gradually-decreasing value of the star-ratio for successively fainter magnitudes. It is of great importance to have accurate knowledge of the rate at which the star-ratio diminishes, and particularly of the way in which it is related to galactic latitude. It may be hoped that the information may lead to a more precise knowledge of the flattening of the stellar system towards the plane of the Milky Way.

The value of any compilation of star-counts will depend mainly on the accuracy with which the magnitudes, to which they refer, have been determined. In modern researches, the standardising of the counts by a special photometric investigation is a *sine quâ non.* But it is only recently that this refinement has come within practical possibilities; and much statistical matter that has been used up to now depends on the ingenious adaptation and correction of data, which were initially rather un-

suitable. It must be admitted that these early researches accomplished their end in the main; and that they have not only prepared the way for more satisfactory determinations, but also have taught us much with regard to stellar distribution that has a lasting value. But having now available sufficient statistics based on sound magnitude-standards, we shall not need to recur to the pioneer discussions, except where discrepancies of particular interest arise.

An investigation of the number of stars of each magnitude by S. Chapman and P. J. Melotte,[1] published in 1914, contains by far the most comprehensive treatment of this problem, and we shall attach the greatest weight to it. The magnitudes are photographic magnitudes based on the Harvard Standard North Polar Sequence. The general accuracy of the Harvard magnitude-scale has been confirmed by investigations made at Mount Wilson and Greenwich; and for our present purposes it is believed to be accurate enough; it is possible, however, that the corrections may not be altogether negligible in future more elaborate discussions. The statistics given by Chapman and Melotte extend from magnitude $2^{m}\cdot 0$ to $17^{m}\cdot 0$. This huge range (representing a light-ratio of 1,000,000 to 1) is filled in almost continuously by data derived from five separate investigations. As each investigation is particularly strong near the middle point of its range, there are five well-determined points. These alone should suffice to give a correct idea of the course of the star-numbers throughout the fifteen magnitude intervals, even without the weaker results, which bridge the gaps.

The five sources of data are as follows:—

(1) Magnitude 12 to 17·5. Counts on the Franklin-Adams chart of the sky. These contain the results from 750 areas scattered over the northern hemisphere, each containing from 60 to 90 stars in all. This represents only a portion of the counting of the Franklin-Adams

chart carried out at Greenwich; but for the other areas the comparison with the standard sequence has not yet been effected, and, accordingly, the results are not used.

(2) Magnitude 9 to 12·5. Counts of stars in the Greenwich Astrographic Catalogue (Dec. + 64° to + 90°). For 195 plates the formulæ for reducing the measured diameter, published in the catalogue, to magnitude had been determined by rigorous comparison with the standard sequence, and these results were used for the present investigation.

(3) Magnitude 6·5 to 9. Counts of stars in the Greenwich Catalogue of Photographic Magnitudes of Stars brighter than $9^{m}{\cdot}0$ between Declination + 75° and the Pole. These magnitude-determinations were made from plates specially taken for the purpose with a portrait-lens, the Astrographic plates being unsuitable for stars of this brightness.

(4) Magnitude 5 to 7·5. Counts of stars in Schwarzschild's Göttingen Actinometry for Dec. 0° to + 20°. A small correction ($0^{m}{\cdot}13$) was required to reduce from the Göttingen to the Harvard scale of magnitudes.

(5) Magnitude 2·0 to 4·5. Counts of stars in the Harvard catalogue of photographic magnitudes of bright stars (*Harvard Annals*, Vol. 71, Pt. I.). This is the least satisfactory part of the data, for the magnitudes were not determined photographically, but were found from the visual magnitudes by applying the colour index corresponding to the known spectral type of each star. The magnitudes were published before the standard sequence appeared, and it is not clear how far they conform to that scale.

As the principal feature in the apparent distribution of the stars is the variation with galactic latitude, the data have been arranged in eight galactic belts. The first seven belts are from 0°—10°, 10°—20°, ... 60°—70°, and belt VIII is from 70°—90° galactic latitude (North or

South). The sources of data (1), (4), (5) cover the whole range, but (2) and (3) are confined to the belts I—V

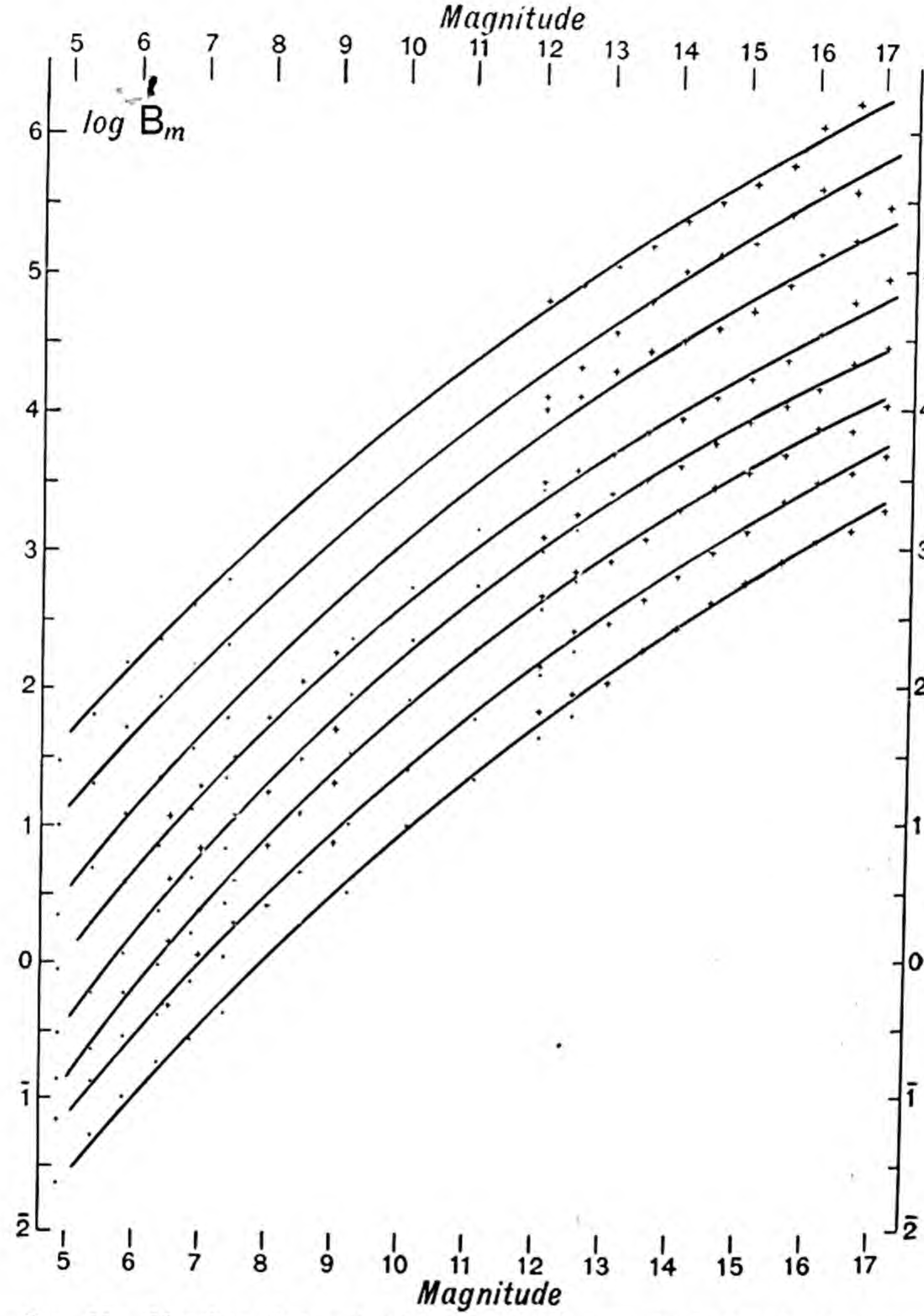

FIG. 20.—Number of stars brighter than each magnitude for eight zones. (Chapman and Melotte).

and II—V respectively; thus the information is not so complete for the three highest zones as for the remainder.

If B_m is the number of stars per square degree brighter than the magnitude m, the counts are most conveniently exhibited by plotting $\log B_m$ against m. This is done for each of the eight belts in Fig. 20, and smooth curves have been drawn to represent the results. In order to prevent overlapping of the eight curves, they have been displaced successively through an amount 0·5 in the vertical direction. The lowest curve corresponds to belt I. The alternation of dots and crosses serves to differentiate the four sources of data. The data (5) are not shown.

It is the central part of the data from each source that is best determined, and the outer parts may be expected to run off the curve a little. The general agreement of the separate sets of data is very satisfactory.

The values of $\log B_m$ for each magnitude, as read from the curves, are given in Table 31.

TABLE 31.

Log B_m for each Magnitude (Chapman and Melotte).

Zone.	I.	II.	III.	IV.	V.	VI.	VII.	VIII.	Whole Sky.
Galactic Latitude.	0°–10°	10°–20°	20°–30°	30°–40°	40°–50°	50°–60°	60°–70°	70°–90°	0°–90°
Magnitude m.									
5·0	$\bar{2}$·435	$\bar{2}$·360	$\bar{2}$·170	$\bar{2}$·055	$\bar{2}$·040	$\bar{2}$·030	$\bar{2}$·105	$\bar{2}$·120	$\bar{2}$·223
6·0	$\bar{1}$·010	$\bar{2}$·950	$\bar{2}$·800	$\bar{2}$·700	$\bar{2}$·655	$\bar{2}$·610	$\bar{2}$·660	$\bar{2}$·680	$\bar{2}$·819
7·0	$\bar{1}$·555	$\bar{1}$·500	$\bar{1}$·385	$\bar{1}$·295	$\bar{1}$·215	$\bar{1}$·170	$\bar{1}$·180	$\bar{1}$·200	$\bar{1}$·377
8·0	0·065	0·015	$\bar{1}$·930	$\bar{1}$·830	$\bar{1}$·730	$\bar{1}$·685	$\bar{1}$·670	$\bar{1}$·670	$\bar{1}$·895
9·0	0·545	0·490	0·420	0·320	0·200	0·165	0·125	0·115	0·374
10·0	0·990	0·935	0·880	0·770	0·640	0·605	0·550	0·520	0·819
11·0	1·405	1·345	1·300	1·180	1·030	1·010	0·940	0·900	1·229
12·0	1·790	1·725	1·680	1·545	1·385	1·385	1·305	1·255	1·605
13·0	2·150	2·075	2·020	1·880	1·715	1·730	1·645	1·585	1·951
14·0	2·485	2·405	2·340	2·185	2·020	2·045	1·965	1·890	2·268
15·0	2·800	2·715	2·630	2·465	2·300	2·335	2·265	2·190	2·575
16·0	3·095	3·005	2·900	2·720	2·565	2·600	2·540	2·470	2·855
17·0	3·380	3·285	3·155	2·965	2·815	2·850	2·810	2·745	3·125

It is of interest to find the total number of stars in the sky, so far as can be deduced from the samples discussed. The results are as follows :—

TABLE 32.

Number of Stars in the Sky brighter than a given Magnitude (Chapman and Melotte).

Limiting Magnitude.	Number of Stars.	Limiting Magnitude.	Number of Stars.
m.		m.	
5·0	689	12·0	1,659,000
6·0	2,715	13·0	3,682,000
7·0	9,810	14·0	7,646,000
8·0	32,360	15·0	15,470,000
9·0	97,400	16·0	29,510,000
10·0	271,800	17·0	54,900,000
11·0	698,000		

The curves in Fig. 20 are approximately parabolic arcs, and the results can be expressed with satisfactory accuracy by empirical formulæ of the type,

$$\log B_m = \alpha + \beta m - \gamma m^2. \quad \quad (1)$$

But it is more convenient to use $m-11$ instead of m, since the zero of magnitude is outside the range we are considering. The formulæ for the eight zones are as follows :—

TABLE 33.

Number of Stars brighter than a given Magnitude.

Zone I. . .	$\log_{10} B_m$	$= 1·404$	$+0·409\ (m-11)$	$-0·0139\ (m-11)^2$
,, II. . .	,,	$= 1·345$	$+0·407$,,	$-0·0147$,,
,, III. . .	,,	$= 1·300$	$+0·411$,,	$-0·0193$,,
,, IV. . .	,,	$= 1·177$	$+0·403$,,	$-0·0186$,,
,, V. . .	,,	$= 1·029$	$+0·391$,,	$-0·0168$,,
,, VI. . .	,,	$= 1·008$	$+0·399$,,	$-0·0160$,,
,, VII. . .	,,	$= 0·941$	$+0·389$,,	$-0·0135$,,
,, VIII. .	,,	$= 0·901$	$+0·380$,,	$-0·0130$,,

The formulæ make it clear that the main part of the variation with galactic latitude consists in a change of the constant term. The coefficient of $(m-11)$ is nearly stationary, and that of $(m-11)^2$ shows no systematic progression with latitude. The ratio of the star-density

near the galactic pole to that near the galactic plane is practically the same for all magnitudes. Or, taking another point of view, the rate of increase in the number of stars with advancing magnitude follows the same law for all latitudes.

This conclusion is, of course, only approximate. We see from Table 31 that the ratio of the star-density in Zone I to that in Zone VIII is—

For magnitude $6^m{\cdot}0$, 2·1 : 1
,, ,, $17^m{\cdot}0$, 4·3 : 1

from which the conclusion may be drawn that the rate of increase in the number of stars is appreciably greater near the galactic plane than away from it. But these differences are very slight compared with those found in some former investigations, which have hitherto found wide acceptance. In particular the celebrated star-gauges of the Herschels, which have dominated our views as to the distribution of faint stars for nearly a century, gave widely different results. Prior to Chapman and Melotte's work the most extensive discussion of magnitude statistics was that of J. C. Kapteyn [2] (1908), which gave a very different idea of the effect of galactic latitude on the star-density. From Kapteyn's table it appears that the ratio of the star-density in Zone I to that in Zone VIII should be—

For magnitude $6^m{\cdot}0$, 2·2 : 1
,, ,, $17^m{\cdot}0$, 45 : 1.

It is difficult to explain the huge difference between Kapteyn's result for the distribution of the faint stars, and that of Chapman and Melotte. The former research was somewhat provisional in character,—an *interim* result to be used until data on a more uniform plan could be obtained; indeed the complete charting of the heavens by J. Franklin-Adams was largely inspired by the influence of Kapteyn, jointly with Sir David Gill, for the purpose of obtaining more satisfactory statistics. Yet the great

divergence between the old and the new is surprising, and it may be well to attempt to trace the precise source from which it arises.

In the first place it must be remarked that Kapteyn's figures for $17^m \cdot 0$ are an extrapolation; his data did not go beyond $14^m \cdot 0$. If then we compare the results for $14^m \cdot 0$, we have to account for a discordance—

Ratio Star-density, Zone I to Zone VIII	Kapteyn	11·5 : 1
,, ,, ,, ,,	Chapman and Melotte	3·9 : 1.

Kapteyn made use of seven main sources of information. Of these the first four (including counts on the plates of the Cape Photographic Durchmusterung) relate to stars brighter than $9^m \cdot 25$; these show no important divergence in galactic distribution from the new figures. For the fainter stars the main reliance was placed on Sir John Herschel's star-gauges,[3] *i.e.*, counts of stars visible with his 18-inch reflector in a field of definite area. Although these are confined to the southern hemisphere, they are more evenly distributed and more typical of normal parts of the heavens than those of Sir William Herschel, and are therefore to be preferred. Arranged according to galactic latitude these gauges give a star-density falling steadily from 1375 stars per square degree at the galactic circle to 137 at the galactic pole. The limiting magnitude is determined by indirect means to be 13·9, and it will be seen that the ratio 10 : 1 is practically equivalent to the definitive result adopted by Kapteyn.

It appears, then, that the Herschel gauges are the main source of the discrepancy; but the results were not accepted without careful checks being applied. These were of two kinds; first, counts of stars were made on forty-five photographs, chiefly of the fields of variable stars, for which the limiting magnitude on a visual scale could be calculated from standard stars, photometrically determined. These, when arranged according to galactic latitude, gave results

in excellent accordance with the star-gauges; and it was considered that this checked the constancy of Herschel's limiting magnitude. The remaining source of statistics was provided by the published charts of the *Carte du Ciel* taken at Algiers, Paris, and Bordeaux. There is no means as yet of determining the limiting magnitude of these independently; but, as it seems reasonable to assume that any fluctuations will be accidental and have no systematic relation to galactic latitude, they may be used to obtain the ratio of galactic concentration. The ratio found by Kapteyn between the belts of latitude $0°-20°$ and $40°-90°$ was 5·5 : 1. For Zones I and VIII the ratio would naturally be greater, and, moreover, the numbers refer to a limit about one magnitude brighter than Herschel's gauges. The ratio 10 : 1 at magnitude 14 is therefore supported by three independent sources of evidence—Sir J. Herschel's gauges, counts on variable star fields, and counts on the French astrographic charts.

It has been suggested by H. H. Turner * that the discordance is due to a real difference in the distribution according as the magnitudes are reckoned visually or photographically. The Herschel counts refer directly to visual magnitudes, and the counts of variable star fields, although made on photographs, are reduced in such a way that the results refer to the visual scale. The counts on the French chart-plates, however, relate solely to the photographic scale; and it is only by disregarding this evidence that the suggestion reconciles the two results. Perhaps we may consider the third source more doubtful than the two former, for the plates are distributed only through a narrow zone, and may be affected by abnormal regions of the sky. The possibility of a real difference in the galactic concentration for visual and photographic results is an interesting one. It appears to mean that

* Who had also arrived at a small value of the galactic concentration for photographic magnitudes (*Monthly Notices*, Vol. 72, p. 700).

there are in the galactic regions vast numbers of faint stars too red to be shown on the photographs. This may be due either to a special abundance of late type stars in the more distant parts of the stellar system, or more probably to the presence of absorbing material—a fog—in interstellar space, which scatters the light of short wave-length.

The hypothesis requires further confirmation. A discussion by E. C. Pickering is directly opposed to it. He also used the visual determinations of magnitude in the fields of variable stars for his data; but he applied them more directly. His conclusion was that "the number of stars for a given area in the Milky Way is about twice as great as in the other regions and the ratio does not increase for faint stars down to the twelfth magnitude[4]."

One of the most interesting results of Chapman and Melotte's investigation is that the total number of stars in the sky for the fainter magnitudes is much smaller than has often been supposed. Kapteyn's table gave 389 million down to $17^{m}\cdot 0$ against 55 million according to the present investigation. The excess of Kapteyn's numbers is almost wholly due to his high value of the galactic concentration; the two investigations are practically in agreement at the galactic poles.

By inspection of Table 32 it will be seen that the rate of increase in the total number of stars has fallen off very considerably for the last few magnitudes included. It appears that the numbers are beginning to approach a limit. An attempt to determine this limit involves a somewhat risky extrapolation, yet the convergence has already become sufficiently marked to render such extrapolation not altogether unjustifiable. The empirical formula $\log B_m = \alpha + \beta m - \gamma m^2$ cannot be pressed far beyond the range for which it was determined, since it leads to the impossible result that the number of stars down to a

given magnitude would ultimately begin to diminish. By a simple modification more suitable formulæ can be obtained. Instead of using B_m we consider $b_m = \frac{dB_m}{dm}$, that is to say, we use the number of stars of the magnitude m instead of the number brighter than m. It is found that an equally good approximation is obtained by setting

$$\log_{10} b_m = a + bm - cm^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

and with this form the whole number of stars approaches asymptotically to a definite limit as m is increased.

We have then

$$\frac{dB_m}{dm} = b_m = 10^{a + bm - cm^2}$$

$$B_m = \int_{-\infty}^{m} e^{(a+bm-cm^2)/\log_{10}e}\, dm$$

$$= \frac{A}{\sqrt{\pi}} \int_{-\infty}^{B(m-C)} e^{-x^2}\, dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where

$$A = \sqrt{\frac{\pi \log_{10}e}{c}} \cdot 10^{a+b^2/4c} \qquad B = \sqrt{\frac{c}{\log_{10}e}} \qquad C = \frac{b}{2c}$$

From the formula (3) it is seen that A represents the total number of stars of all magnitudes, and C represents the median magnitude, *i.e.*, the limit to which we must go to include half the stars.

As c is not very easy to determine, two formulæ may be given between which the truth probably lies :—

$$\log_{10} b_m = -0{\cdot}18 + 0{\cdot}720\, m - 0{\cdot}0160\, m^2 \quad . \quad . \quad (4)$$
$$\log_{10} b_m = +0{\cdot}01 + 0{\cdot}680\, m - 0{\cdot}0140\, m^2 \quad . \quad . \quad (5)$$

We have here altered the unit of area, so that b_m refers to the number of stars in the whole sky instead of to a square degree.

These formulæ lead to

	(4)	(5)
Whole number of stars of all magnitudes	**770 millions**	**1800 millions**
Median magnitude	**$22^{m}{\cdot}5$**	**$24^{m}{\cdot}3$**

Chapman and Melotte conclude : " Unless the general form of our expression for B_m ceases to apply for values

of m greater than 17 (up to which the accordance is good) it is possible to say with some probability that half the total number of stars are brighter than the 23rd or 24th magnitude, and that the total number of stars is not less than one thousand millions and cannot greatly exceed twice this amount."

The mean star-density for given galactic latitude and limiting magnitude has been given in Table 31. A question arises as to how far these are sufficient to determine the star-density at any particular spot, and what variations from the mean are likely to arise. The possible variations may be classed as follows.

(1) A systematic difference between the north and south galactic hemispheres.

(2) A systematic dependence on galactic longitude in some of the zones.

(3) General irregularity.

There is not a very conspicuous difference between the two galactic hemispheres. So far as the tenth magnitude the southern hemisphere is found to be the richer by 10 or 15 per cent. ; for fainter stars no difference is found. We have to depend for this conclusion on researches made before the introduction of modern standard magnitudes, but the evidence seems to be satisfactory. To account for the small difference in richness it has been usual to suppose that the Sun is a little north of the central plane of the stellar system; this agrees with the appearance of the Milky Way, which deviates slightly from a great-circle, and has a mean S. Galactic Latitude of 1°·7.

It is the opinion of most investigators that, except in the Milky Way itself, the differences depending on galactic longitude are inconsiderable; galactic latitude is the one important factor and overshadows all other variations. This view seems to be based on a general impression rather than on any quantitative results. A

detailed investigation is greatly to be desired, for the conclusion must still be considered open to question. The following calculation appears to furnish an upper limit to the possible variations of the limiting magnitude.

For the majority of the Franklin-Adams chart-plates, unstandardised counts have been given by Chapman and Melotte. Taking, for example, the Johannesburg plates, the centres of which lie between galactic latitudes 20°—29° and 30°—39°, we find the following results, which refer to a limiting magnitude about $17^{m}\cdot5$:—

Zone.	20°—29°	30°—39°
Number of Plates	20	16
Smallest number of stars per plate . .	292,000	306,000
Largest ,, ,, ,, ,, . .	737,000	577,000
Average deviation of log density from the mean	±0·078	±0·059
Corresponding ratio	1·20 : 1	1·15 : 1

The average deviation (20 and 15 per cent. respectively) includes not only the actual fluctuations of star-density, but variations due to the quality of the plates and the personality of the counters. Its smallness bears witness to the uniformity of the Johannesburg sky as well as to the regularity of stellar distribution.

There is no doubt that within the limits of the Milky Way very considerable variations of star-density occur. The most remarkable region is in the constellation Sagittarius, where certain of the star-clouds are extraordinarily rich. This part of the Milky Way is unfavourably placed for observation in the latitude of the British Isles; but from more southerly stations it appears the most striking feature of the heavens. It was found that on the Franklin-Adams plates the images of the faintest stars in this region were so close as to merge into a continuous background and it was impossible to count them. We should expect that the presence of the Milky Way clusters would add great numbers of stars beyond the normal increase towards the galactic plane, and it is

rather surprising that there is not a more marked discontinuity between the numbers for Zone I and Zone II in Table 31. Probably the dark spaces and tracts of absorbing matter, which are a feature of the Milky Way, neutralise the effect of the rich regions, and bring about a general balance.

The star-ratio falls a great deal below its theoretical value 3·98 for an infinite universe even as early as the sixth magnitude. The magnitude-counts are not, however, sufficient by themselves to determine the rate at which the stars thin out at great distances. For this we need additional statistics of a different kind, as will be shown in the next chapter. Meanwhile, although the star-counts do not determine a definite stellar distribution, we can examine whether any simple form of the law of stellar density in space is consistent with them. The simple result that the galactic concentration is independent of the magnitude, even if it were rigorously true, would not admit of any correspondingly simple interpretation in terms of the true distribution in space. It is therefore of no help in our discussion.

Consider a stellar system in which the surfaces of equal density are similar and similarly situated with respect to the Sun as centre, the density falling off from the inner parts to the outer. We naturally think of spheroids of the same oblateness.

Let the ratio of the radii towards the galactic pole and in the galactic plane be $1 : \nu$.

Corresponding to an element of volume S at a distance r towards the pole, there will be an element of volume $\nu^3 S$ at a distance νr in the galactic plane, containing stars distributed with the same density. The number of these stars, being simply proportional to the volume, will be ν^3 times as many in the second case, and this will apply to all grades of intrinsic brightness separately. But their

apparent brightness will be diminished in the ratio ν^{-2}, or, expressed in magnitudes, they will be 5 log ν magnitudes fainter. This holds for all the elements of volume S in a cone from the Sun to the galactic pole, and the corresponding elements $\nu^3 S$ in a cone from the Sun to the galactic equator.

Hence if the number of stars brighter than a given magnitude is given by

$$B_m = \psi(m) \text{ for the galactic pole}$$

it will be given by

$$B_m = \nu^3\psi(m - 5\log_{10}\nu) \text{ for the galactic plane.}$$

Now we have found (Table 33) that for the galactic pole

$$\log B_m = 0{\cdot}901 + 0{\cdot}380\,(m - 11) - 0{\cdot}013\,(m - 11)^2.$$

Hence for the galactic plane

$$\log B_m = 3\log\nu + 0{\cdot}901 - 0{\cdot}380 \times 5\log\nu - 0{\cdot}013\,(5\log\nu)^2 + (0{\cdot}380 + 0{\cdot}013 \times 10\log\nu)\,(m - 11) - 0{\cdot}013\,(m - 11)^2.$$

If $\log\nu = 0{\cdot}54$, this reduces to

$$\log B_m = 1{\cdot}400 + 0{\cdot}450\,(m - 11) - 0{\cdot}0130\,(m - 11)^2,$$

which is fairly close to the true value for Zone I, viz.

$$\log B_m = 1{\cdot}404 + 0\,409\,(m - 11) - 0{\cdot}0139\,(m - 11)^2.$$

The excess of the calculated coefficient 0·450 over its observed value can be interpreted to mean that the diminution of density in the galactic plane is rather more rapid than would be the case if the surfaces of equal density were similar. The oblateness of the distribution becomes less pronounced at great distances. The difference is not due to accidental error, for it can be shown that any other pair of zones would have given a greater excess in the same sense. The oblateness ν is therefore not a constant quantity; and the above value corresponds to a certain average distance which may be roughly expressed as the distance of the stars of the eleventh magnitude.

For $\log \nu = 0{\cdot}54$, we have $\nu = 3{\cdot}5$. This, then, is the average oblateness, or ratio of the axes of the surfaces of equal density. ν must not be confused with the galactic concentration of the stars; it is a pure accident that their values are nearly the same.

REFERENCES.—CHAPTER IX.

1. Chapman and Melotte, *Memoirs, R.A.S.*, Vol. 60, Pt. 4.
2. Kapteyn, *Groningen Publications*, No. 18.
3. J. Herschel, *Results of Astronomical Observations at the Cape of Good Hope*, Cp. iv.
4. Pickering, *Harvard Annals*, Vol. 48, p. 185.

CHAPTER X

GENERAL STATISTICAL INVESTIGATIONS

In the application of mathematics to the study of natural phenomena, it is necessary to treat, not the actual objects of nature, but idealised systems with a few well-defined properties. It is a matter for the judgment of the investigator, which of the natural properties shall be retained in his ideal problem, and which shall be cast aside as unimportant details; he is seldom able to give a strict proof that the things he neglects are unessential, but by a kind of instinct or by gradual experience he decides (sometimes erroneously it may be) how far his representation is sufficient.

In the idealised stellar system, which will now be considered, there are three chief properties or laws. The determination of these must be regarded as the principal aim of investigations into the structure of the sidereal universe, for if they were thoroughly known we might claim a very fair knowledge of the distribution and movements of the stars. The laws are :—

(1) The *density* law.—The number of stars per unit volume of space in different parts of the system.

(2) The *luminosity* law.—The proportion of stars between different limits of absolute brightness.

(3) The *velocity* law.—The proportion of stars having linear velocities between different limits both of amount and direction.

As regards the first of these, the density may be assumed to depend on the distance from the Sun and on the galactic latitude. The decrease of density at great distances from the Sun represents the fact that the stellar system is limited in extent, and as it is notorious that the limits are very much nearer towards the galactic poles than in the galactic plane, a representation which did not include a variation with galactic latitude would be very imperfect.

The luminosity law and velocity law may be assumed tentatively to be the same in all parts of space. Arguments may be urged against both these assumptions; but they seem to be inevitable in the present state of knowledge, and it is probable that the results obtained on this basis will be valid as a first approximation. It may further be remarked that in dealing with proper motions we are necessarily confined to a rather small volume of the stellar system, and the assumption of a constant velocity law in such investigations seems to be justified.

The constancy of the velocity law is in most investigations assumed in a different form, which must be carefully distinguished from the assumption just stated, and is, in fact, much less innocuous. It is assumed that for the stars *of a catalogue* the velocity law is the same at all distances. Now among the stars of a catalogue with a lower limit of magnitude there is a strong correlation between luminosity and distance. Thus an additional assumption is virtually made, that the stars of different intrinsic luminosity have the same velocity law; or, since spectral type and brightness are closely associated, that the stars of different spectra have the same velocity law. This is well-known to be untrue. It seems likely that results obtained on this assumption (including some investigations in this chapter) may be misleading in some particulars; though here again we may often arrive at fairly correct conclusions notwithstanding imperfect methods. But it is desirable, where possible, that the

different spectral types should be investigated separately; for in the case of stars of homogeneous type we know of no evidence to invalidate the hypothesis of a constant velocity law.

A further property of the stellar system, which has some claim to be retained in the idealised representation, is the absorption of light in inter-stellar space. There is some evidence, insufficient it may be, that this is small enough to be neglected in the present discussions. As it does not seem practicable to deduce useful results when it is retained as an unknown, we shall take the risk of neglecting it.

The problem of determining one or more of the three laws that have been enumerated may be attacked in various ways; and the diversity of the investigations is rather bewildering. The difficulty of giving a connected account of the present state of the problem is increased by the fact that some of the work has been based on data that are now obsolescent; and it is difficult to know how far the introduction of more recent figures would produce important modifications.

The general statistical researches described in this chapter depend on one or more of the following classes of data:—

(*a*) Counts of stars between given limits of magnitude.

(*b*) The mean parallactic motion of stars of given magnitudes.

(*c*) Parallaxes measured directly.

(*d*) The observed distribution (or spread) of the proper motions of stars brighter than a limiting magnitude.

The radial velocities are only appealed to for the adopted speed of the solar motion, usually taken as 19·5 km. per sec.

It is convenient to distinguish the use of the proper motions for determining the mean parallactic motion from other applications of the proper motion data. The parallactic motion, or as it is sometimes called, the secular

parallax, fixes the mean distance of a class of stars without introducing any consideration of the distribution of their individual velocities.

The investigations may be grouped into three classes:—

I. Those which depend on (*a*) and (*b*) only.
II. Those which depend on (*a*), (*b*), (*c*), and (*d*).
III. Those which depend on (*d*) only.

It will be shown later that (*a*) and (*b*) are theoretically sufficient to determine the density and luminosity laws, so that the inclusion of (*c*) introduces some redundancy of equations. Investigations depending on (*d*) stand rather apart from the others, since they involve the velocity law; but as they also throw some light on the other two laws, it is useful to consider them in the same connection.

Before proceeding to consider the three classes of investigation, it is necessary to give some attention to the expression of the results for the mean parallactic motions and the measured parallaxes in a suitable form. The formulæ given by J. C. Kapteyn in 1901 are very widely used; and, although it would naturally be an improvement to substitute more recent data, no general revision of his work has yet been made. Kapteyn derived two formulæ for the mean parallaxes of stars; one expressing the mean for all stars of a given magnitude, the other for stars of given magnitude and proper motion. For the first formula it is not possible to make use of the directly measured parallaxes. It is not so much that the data are too scanty, as that the stars are usually selected for investigation on account of large proper motion, and are accordingly much nearer to the Sun than the bulk of the stars of the same magnitude. The mean parallactic motion, or secular parallax, provides the necessary means of determining the dependence of parallax on magnitude alone. The only practical difficulty arises from the occasional excessive motions, which exercise a dispropor-

tionately great influence on the result. Kapteyn's results, which depend on the Auwers-Bradley proper motions, are contained in the formula

$$\text{Mean parallax for magnitude } m = 0''{\cdot}0158 \times (0{\cdot}78)^{m-5{\cdot}5} \quad . \quad . \quad \text{(A)}$$

If Types I and II are taken separately, the formulæ for the mean parallaxes are :

$$\text{Type I} \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \bar{\pi}_m = 0''{\cdot}0097 \times (0{\cdot}78)^{m-5{\cdot}5}.$$
$$\text{,, II} \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \bar{\pi}_m = 0''{\cdot}0227 \times (0{\cdot}78)^{m-5{\cdot}5}.$$

In Table 34 the mean parallaxes for the different magnitudes are given, after correcting the foregoing formula (A) to reduce to the modern value of the solar motion, 19·5 km. per sec., instead of 16·7 used by Kapteyn.

TABLE 34.

Kapteyn's Mean Parallaxes.

(Reduced to the value 19·5 km. per sec. for the solar motion.)

Magnitude.	Mean Parallax.	Magnitude.	Mean Parallax.
	"		"
1·0	0·0414	6·0	0·0120
2·0	0·0323	7·0	0·0093
3·0	0·0252	8·0	0·0073
4·0	0·0196	9·0	0·0057
5·0	0·0153		

For the dependence of parallax on proper motion, Kapteyn had recourse to the measured parallaxes. For this purpose their use is quite legitimate, though we may be inclined to doubt whether the data (at that time much less satisfactory than now) were sufficiently trustworthy. For a constant magnitude, the dependence on the proper motion μ was found to follow the empirical formula

$$\bar{\pi} \propto \mu^p$$

where $p = 1/1{\cdot}405$. The actual formula giving the

mean parallax of a star of magnitude m and proper motion μ'' per annum is

$$\bar{\pi}_{m,\mu} = (0\cdot905)^{m-5\cdot5} \times (0\cdot0387\mu)^{0\cdot712} \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(B)}$$

It is interesting to know how nearly this mean formula is likely to give the correct parallax of a particular star. Assuming that $\log(\pi/\bar{\pi})$ is distributed according to the law of errors, the probable deviation of this logarithm has been found to be 0·19. Thus it is an even chance that the parallax of any star will lie between 0·65 and 1·55 times the most probable value for the given motion and magnitude.* This result was found by comparing measured parallaxes with the formula, and determining the average residual.

The formulæ for $\bar{\pi}_m$ and $\bar{\pi}_{m,\mu}$ may be more conveniently expressed in the form †

$$\log_{10}\bar{\pi}_m = -1\cdot108 - 0\cdot125m \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(C)}$$

$$\log_{10}\bar{\pi}_{m,\mu} = -0\cdot766 - 0\cdot0434m + 0\cdot712\log\mu \quad . \; . \; . \; . \; . \; . \; . \quad \text{(D)}$$

In order to bring together all the data due to Kapteyn, we may add here his results for the counts of stars of successive magnitudes. It has been shown by K. Schwarzschild that Kapteyn's numbers can be summarised by the formula:

$$\log_{10}b_m = 0\cdot596 + 0\cdot5612\,m - 0\cdot0055\,m^2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(E)}$$

where b_m is the number of stars (in the whole sky) between magnitude m and $m + dm$.

The formulæ (C), (D), and (E) correspond respectively to the data (b), (c), and (a) previously mentioned.

The main criticism of subsequent investigators has been directed against the large value of the coefficient of m in formula (C). There is some reason to believe that the decrease of mean parallax with increasing faintness is less rapid than is shown in Table 34. According to C. V.

* The most probable value is not the mean value. With the above probable deviation the most probable value would be $0\cdot81 \times \bar{\pi}_{m,\mu}$.

† Formulæ (A) and (C) do not quite correspond as the former contains a slight correction made by Kapteyn (Preface, *Groningen Publications*, No. 8).

L. Charlier[1] the coefficient is less than half Kapteyn's value. Charlier's determination rests on the Boss proper motions, which are of great accuracy; but the range of magnitude is too limited for a satisfactory solution. Without attaching much weight to the precise value, he considers that Boss's proper motions cannot be reconciled with the larger figure. This means that there is less difference between the parallaxes of different magnitudes than Kapteyn supposed. The writer,[2] also working on Boss's Catalogue, had found that for stars of a given magnitude the spread in distance must be less than that deduced from Kapteyn's formulæ, a fact which is evidently related to Charlier's objection.

G. C. Comstock[3] has likewise maintained that the parallaxes of faint stars are larger than those given by the formula. From an investigation of the proper motions of 479 stars from 7^m to 13^m he concluded that a relation, first given by A. Auwers for a shorter range, holds satisfactorily from the third to the thirteenth magnitudes, viz., that the mean proper motion is inversely proportional to the magnitude. As the mean proper motion may be taken to be proportional to the parallax, Comstock's result leads to the formula

$$\bar{\pi} = c/m \qquad (m > 3).$$

The formula (E) may be compared with Chapman and Melotte's determination (formulæ (4) and (5), p. 195). The real differences are very large, though perhaps not quite so great as would appear from a cursory comparison of the coefficients.

I. Investigations which Depend on Counts of Stars and on Mean Parallactic Motions.

From the mean parallaxes of stars of different magnitudes combined with the counts of the number of stars down to limiting magnitudes, the density law and the

luminosity law can be determined. A most elegant general solution of this problem has been given by Schwarzschild; and, although it may usually be less laborious in practice to work out special cases according to the functions which represent the observed data, his method is so generally applicable to the fundamental problems of stellar statistics that we shall begin by considering it at length.

Let $D(r)$ be the number of stars per unit volume at a distance r from the Sun.

Let $\phi(i)\,di$ be the proportion of which the absolute luminosity lies between i and $i+di$.

Setting h for the apparent brightness of a star, we have

$$h = \frac{i}{r^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Let $B(h)\,dh$ be the total number of stars of apparent brightness between h and $h+dh$.

And let $\pi(h)$ be the mean parallax of stars of apparent brightness h.

Then the whole number of stars at distances between r and $r+dr$ is

$$4\pi r^2\,dr \,.\, D(r),$$

and of these the proportion $\phi(hr^2)\,r^2\,dh$ will have an apparent brightness between h and $h+dh$.

Hence

$$B(h)\,dh = \int_{r=0}^{\infty} 4\pi r^2\,dr\,D(r)\,\phi(hr^2)\,r^2\,dh,$$

or

$$B(h) = 4\pi \int_0^{\infty} D(r)\,\phi(hr^2)\,r^4\,dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

And for the sum of the reciprocals of the distances

$$B(h)\,\pi(h) = 4\pi \int_0^{\infty} D(r)\,\phi(hr^2)\,r^3\,dr \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

We now transform the two integrals (2) and (3) as follows:—

Let

$$r = e^{-\rho}, \quad h = e^{-2\mu},$$

so that

$$i = e^{-2(\mu+\rho)}.$$

Further let

$$\begin{aligned} 4\pi\, D(r)\, r^{5} &= f(\rho), \\ \phi(i) &= g(\mu+\rho), \\ B(h) &= b(\mu), \\ B(h)\,\pi(h) &= c(\mu). \end{aligned}$$

Here, since $B(h)$ and $\pi(h)$ are supposed to be given by the data of observation, $b(\mu)$ and $c(\mu)$ are given likewise; they are the same observed quantities expressed as functions of a changed independent variable.

The two integral equations (2) and (3) become

$$b(\mu) = \int_{-\infty}^{\infty} f(\rho)\, g(\mu+\rho)\, d\rho \quad \dots \dots \quad (4)$$

$$c(\mu) = \int_{-\infty}^{\infty} f(\rho)\, g(\mu+\rho)\, e^{\rho}\, d\rho \quad \dots \dots \quad (5)$$

Let us form the Fourier integrals

$$\left.\begin{aligned} \mathfrak{b}(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\mu)\, e^{-\iota q\mu}\, d\mu & \mathfrak{c}(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} c(\mu)\, e^{-\iota q\mu}\, d\mu \\ \mathfrak{f}(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\mu)\, e^{-\iota q\mu}\, d\mu & \mathfrak{g}(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\mu)\, e^{-\iota q\mu}\, d\mu \end{aligned}\right\} \quad (6)$$

where $\iota = \sqrt{-1}$.

Then we have the well-known reciprocal relation

$$\left.\begin{aligned} b(\mu) &= \int_{-\infty}^{\infty} \mathfrak{b}(q)\, e^{\iota q\mu}\, dq & c(\mu) &= \int_{-\infty}^{\infty} \mathfrak{c}(q)\, e^{\iota q\mu}\, dq \\ f(\mu) &= \int_{-\infty}^{\infty} \mathfrak{f}(q)\, e^{\iota q\mu}\, dq & g(\mu) &= \int_{-\infty}^{\infty} \mathfrak{g}(q)\, e^{\iota q\mu}\, dq \end{aligned}\right\} \quad (7)$$

Now

$$\begin{aligned} \mathfrak{b}(q) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} b(\mu)\, e^{-\iota q\mu}\, d\mu \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\rho)\, g(\mu+\rho)\, e^{-\iota q\mu}\, d\mu\, d\rho. \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\rho)\, e^{\iota q\rho}\, d\rho \int_{-\infty}^{\infty} g(\mu+\rho)\, e^{-\iota q(\mu+\rho)}\, d(\mu+\rho). \end{aligned}$$

Hence by (6)

$$\mathfrak{b}(q) = 2\pi\, \mathfrak{f}(-q)\, .\, \mathfrak{g}(q) \quad \dots \dots \quad (8)$$

Similarly,

$$\mathfrak{c}(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\rho)\, e^{\rho}\, e^{\iota q\rho}\, d\rho \int_{-\infty}^{\infty} g(\mu+\rho)\, e^{-\iota q(\mu+\rho)}\, d(\mu+\rho)$$

$$= \mathfrak{g}(q)\int_{-\infty}^{\infty} f(\rho)\, e^{-\iota(\iota - q)\rho}\, d\rho$$

Thus

$$\mathfrak{c}(q) = 2\pi\, \mathfrak{f}(\iota - q)\, \mathfrak{g}(q) \qquad (9)$$

From (8) and (9) we have

$$\frac{\mathfrak{f}(\iota - q)}{\mathfrak{f}(-q)} = \frac{\mathfrak{c}(q)}{\mathfrak{b}(q)} \qquad (10)$$

As the functions $\mathfrak{c}$ and $\mathfrak{b}$ can be calculated directly (by Fourier analysis or otherwise) from c and b, the right-hand side is known.

Setting

$$p = \iota q$$

and

$$F(p) = \log \mathfrak{f}(\iota p) = \log \mathfrak{f}(-q)$$

equation (10) becomes.

$$F(p+1) - F(p) = \log\frac{\mathfrak{c}(-\iota p)}{\mathfrak{b}(-\iota p)} \qquad (11)$$

a difference equation of which the solution is

$$F(p) = \frac{\iota}{2}\int_{-\iota\infty}^{+\iota\infty} \log\frac{\mathfrak{c}(-\iota p')}{\mathfrak{b}(-\iota p')} \cot(p'-p)\pi\, dp' \qquad (12)$$

the integration being along the imaginary axis of p'.

Thus F can be found, and from it $\mathfrak{f}$. Then f is determined by means of (7).

Further, when $\mathfrak{f}$ has been determined, $\mathfrak{g}$ is given by (8) and then g can be determined.

The density and luminosity laws are accordingly found.

If particular forms are assumed for the expressions which give the counts of stars of each magnitude and their mean parallaxes, the analysis may be made much simpler. In the following investigation special forms of the functions are adopted, which appear adequate to represent the present

state of our knowledge, and lend themselves conveniently to mathematical treatment :—

Let

r be the distance measured in parsecs,
i the absolute luminosity measured in terms of a star of zero magnitude at a distance of one parsec,
h the apparent brightness in terms of a star of zero-magnitude.

And set

$$\rho = -5{\cdot}0 \log_{10} r,$$
$$M = -2{\cdot}5 \log_{10} i,$$
$$m = -2{\cdot}5 \log_{10} h,$$

where M and m are accordingly the absolute and apparent magnitudes.

We have $i = hr^2$, and $M = m + \rho$.

We adopt the forms :

$$\text{Density law : } D(r) = 10^{a_0 - a_1\rho - a_2\rho^2} \quad \text{(13)}$$
$$\text{Luminosity law : } \phi(i) = 10^{b_0 - b_1 M - b_2 M^2} \quad \text{(14)}$$

Number of stars between magnitudes m and $m + dm$:

$$b(m)\,dm = 10^{\kappa_0 - \kappa_1 m - \kappa_2 m^2}\,dm \quad \text{(15)}$$

Mean parallax of stars of magnitude m

$$\pi(m) = 10^{p_0 - p_1 m - p_2 m^2} \quad \text{(16)}$$

Then, expressing that the stars of magnitude m to $m + dm$ are made up from successive spherical shells at distance r, containing $4\pi r^2\,dr\,D(r)$ stars, of which the proportion $\phi(hr^2)\,r^2\,dh$ are of the appropriate intrinsic brightness, we have

$$b(m)\,dm = 4\pi \int_0^\infty D(r)\,\phi(hr^2)\,r^4\,dr\,dh \quad \text{(17)}$$

Now

$$dm = -2{\cdot}5 \log_{10} e\; dh/h,$$
$$d\rho = -5{\cdot}0 \log_{10} e\; dr/r,$$
$$r = 10^{-0{\cdot}2\rho},$$
$$h = 10^{-0{\cdot}4m},$$

and

$$\phi(hr^2)\,D(r) = 10^{b_0 - b_1(m+\rho) - b_2(m+\rho)^2 + a_0 - a_1\rho - a_2\rho^2}.$$

Hence we have $b(m) =$

$$\frac{4\pi}{12\cdot5\,(\log e)^2}\int_{-\infty}^{\infty} d\rho \,.\, 10^{-\rho-0\cdot4\,m+a_0-a_1\rho-a_2\rho^2+b_0-b_1(m+\rho)-b_2(m+\rho)^2}. \quad (18)$$

Now the integral is of a well-known form,

$$\int_{-\infty}^{\infty} d\rho\, 10^{A_0-A_1\rho-A_2\rho^2} = \sqrt{\frac{\pi\log e}{A_2}}\,10^{A_0+A_1^2/4A_2}.$$

The reduction evidently leads to an expression for $b(m)$ of the form set down in (15), and we find

$$\left.\begin{aligned}\kappa_0 &= 0\cdot7942-\tfrac{1}{2}\log(a_2+b_2)+a_0+b_0+\frac{(1+a_1+b_1)^2}{4(a_2+b_2)}\\ \kappa_1 &= \frac{a_2(b_1+0\cdot4)-b_2(a_1+0\cdot6)}{a_2+b_2}\\ \kappa_2 &= \frac{a_2b_2}{a_2+b_2}\end{aligned}\right\} \quad \ldots\ldots \quad (19)$$

The sum of the parallaxes of stars between magnitudes m and $m+dm$, which is equal to $\pi(m)\,b(m)\,dm$, is found by writing r^3 for r^4 in (17), or $\pi(m)\,b(m)$ is given by writing $a_1-0\cdot2$ for a_1 in (18) and (19). Carrying through this change in (19), we find

$$\left.\begin{aligned}\kappa_0+P_0 &= 0\cdot7942-\tfrac{1}{2}\log(a_2+b_2)+a_0+b_0+\frac{(0\cdot8+a_1+b_1)^2}{4(a_2+b_2)}\\ \kappa_1+P_1 &= \frac{a_2(b_1+0\cdot4)-b_2(a_1+0\cdot4)}{a_2+b_2}\\ \kappa_2+P_2 &= \frac{a_2b_2}{a_2+b_2}\end{aligned}\right\} \quad \ldots \quad (20)$$

Hence subtracting (19) from (20)

$$\left.\begin{aligned}P_0 &= -0\cdot1\frac{a_1+b_1+0\cdot9}{a_2+b_2}\\ P_1 &= \frac{0\cdot2b_2}{a_2+b_2}\\ P_2 &= 0\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \quad (21)$$

The fact, that quadratic formulæ for the logarithms of the density and luminosity functions lead to a linear formula ($P_2=0$) for the logarithm of the mean parallax is of interest, because the linear formula for the latter is the one given by Kapteyn, and is in general use.

From the formulæ (19) and (21) it is easy to deduce the

coefficients of the density and luminosity functions in terms of the observed coefficients κ_0, κ_1, κ_2, P_0, and P_1. Thus $a_2 = \frac{\kappa_2}{5P_1}$, $b_2 = \frac{\kappa_2}{1-5P_1}$, and so on. a_0 and b_0 are not given independently, but as a sum $a_0 + b_0$. But, if desired, b_0 can be found from the condition implied in the definition of ϕ,

$$\int_0^\infty \phi(i)\, di = 1.$$

Of practical attempts to determine the density and luminosity functions from the mean parallaxes and counts of stars of different magnitudes, an investigation by H. Seeliger [4] (1912) may be taken as an example. Dividing the sky into five zones according to galactic latitude, he obtained for B_m (the number of stars per square degree down to magnitude m) the following expressions:

Zone.	Galactic Latitude.	Formula.
A	$\pm 90°$ to $\pm 70°$	$\log_{10} B_m = -4{\cdot}610 + 0{\cdot}6640\, m - 0{\cdot}01334\, m^2$
B	$\pm 70°$ to $\pm 50°$	,, $= -4{\cdot}423 + 0{\cdot}6099\, m - 0{\cdot}00957\, m^2$
C	$\pm 50°$ to $\pm 30°$	,, $= -4{\cdot}565 + 0{\cdot}6457\, m - 0{\cdot}01025\, m^2$
D	$\pm 30°$ to $\pm 10°$	,, $= -4{\cdot}623 + 0{\cdot}6753\, m - 0{\cdot}01027\, m^2$
E	$+ 10°$ to $- 10°$	,, $= -4{\cdot}270 + 0{\cdot}6041\, m - 0{\cdot}00512\, m^2$

These were derived from a discussion of the magnitudes of the Bonn Durchmusterung and from Sir John Herschel's gauges. Owing to the use of the latter source, the galactic concentration of the faint stars is very strong, just as in Kapteyn's results; the doubt cast on these numbers by modern researches has been fully discussed in the preceding chapter. The course of Seeliger's coefficients from zone to zone appears quite irregular, but the form of the expressions tends to conceal a steady progression in the actual numbers.

From these expressions for B_m, $b_m = \frac{dB_m}{dm}$ was deduced without difficulty and expressed in the same quadratic form (equation 15). For the mean parallaxes (equation

16) Kapteyn's numbers were adopted. But a difficulty arises because the mean parallaxes have been given only for the sky as a whole, and not for the separate zones of galactic latitude; and it is well known that the mean parallax varies greatly from one latitude to another. The difficulty, however, can be surmounted. We have agreed to regard the density law as depending on the galactic latitude, and the luminosity law as constant; that is to say, the coefficients a_0, a_1, a_2 will be different for each zone; b_0, b_1, b_2 will be the same for all. If we eliminate a_1 and a_2 from (19) and (21), and disregard the first equation, which is only used for determining a_0+b_0, we obtain two equations between b_1, b_2 and P_0, P_1 from each zone. Combining these according to the number of stars in each zone, two equations are obtained in which it is permissible to substitute the mean P_1 and P_0 given by Kapteyn's parallaxes. These determine b_1 and b_2. The remaining constants a_0+b_0, a_1, a_2 are then determined for each zone separately by means of equations (19). By a procedure of this kind, though differing in detail, Seeliger arrived at a solution. That the result is very different from what would be obtained by using the same parallax formula for all zones may be seen by the following numbers, which were deduced as the mean parallaxes of stars of magnitude $9^m \cdot 0$.

Zone.	A.	B.	C.	D.	E.
Mean Parallax ($9^m \cdot 0$)	$0''\cdot0065$	$0''0061$	$0''\cdot0053$	$0''\cdot0044$	$0''\cdot0039$

The density and luminosity functions are given by (13) and (14). Seeliger's result for the luminosity function is

$$\phi(i) = \text{const.} \times e^{-2\cdot129 \log_e i - 0\cdot1007 (\log_e i)^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

As examples of the law of density the following (given by Seeliger) may suffice:—

Zone.	A.	B.	C.	D.	E.
$\dfrac{\text{Density at distance 5000 parsecs}}{\text{Density at the Sun}}$	0·0031	0·0049	0·0355	0·0692	0·0851
$\dfrac{\text{Density at 1600 parsecs}}{\text{Density at 16 parsecs}}$	0·021	0·030	0·107	0·166	0·191

These results show a much more rapid falling off in density near the poles compared with the galactic plane. The results for Zone E should perhaps be set aside, since presumably they are disturbed by the passage of the Milky Way through that region; but the other four zones represent the general distribution in the stellar system. We do not, however, place much reliance on the numerical results, since the determination rests on the Herschel gauges and on Kapteyn's mean parallaxes, both of which are open to some doubt.

II. Investigations which Depend on Counts of Stars, the Mean Parallactic Motions, Measured Parallaxes and Distribution of Proper Motions.

Suppose that a table of double-entry is formed giving the number of stars between given limits of magnitude and given limits of proper motion. For the stars in any compartment of the table corresponding to magnitude m and proper motion μ, the formulæ (B) or (D) give the mean parallax. Further, as already stated, the individual parallaxes deviate from the mean according to the law,

Frequency of log $(\pi/\bar{\pi})$ is an error function with probable error 0·19.

Hence the proportion of these stars between any given limits of parallax can be found. Thus, taking the stars in any one compartment, we can redistribute them into a new table with arguments parallax and magnitude. Treating each compartment of the old table separately, we transfer all the stars into the new table, and obtain the number of stars between given limits of magnitude and of parallax.

Table 35 gives the results of a solution made in this way by J. C. Kapteyn.[5] It shows how the number of stars of each magnitude are distributed as regards distance.

TABLE 35.

Distribution in Distance of Stars of each Magnitude (Kapteyn).

Limits of Distance.	Mean Parallax.	Number of Stars in the Sky.						m-M.
Parsecs.	″	m. 2·6–3·6	m. 3·6–4·6	m. 4·6–5·6	m. 5·6–6·6	m. 6·6–7·6	m. 7·6–8·6	m.
>1000	—	0·6	5·0	25	127	703	4840	—
631–1000	0·00118	2·0	8·0	42	197	871	4590	14·5
398–631	0·00187	2·9	19·6	92	369	1466	6050	13·5
251–398	0·00296	9·4	29·6	151	603	2210	7310	12·5
158–251	0·00469	14·7	51·0	223	815	2770	8320	11·5
100–158	0·00743	19·6	64·6	256	885	2760	5830	10·5
63–100	0·0118	22·8	72·8	240	767	2080	4140	9·5
40–63	0·0187	21·3	71·1	190	537	1240	2150	8·5
25–40	0·0296	17·2	57·1	130	311	579	890	7·5
16–25	0·0469	11·8	39·1	71	145	235	316	6·5
10–16	0·0743	6·5	22·5	34	56	84	99	5·5
6·3–10	0·118	3·2	11·2	14	18	29	30	4·5
0–6·3	—	2·0	7·0	8	11	14	14	—

For stars of known magnitude and distance the absolute magnitude M can be calculated. The number to be subtracted from the apparent magnitude m to give the absolute magnitude will be found in the last column of the table. The limits of distance were so chosen that the step from one line to the next corresponds to a change of one magnitude.

Each line of Table 35 gives a determination of the luminosity law, for it exhibits the number of stars in a certain volume of space which have absolute magnitudes between given limits. By taking suitable means between the results from each line of the table, Kapteyn arrived at an expression for the luminosity law which may be written

$$\phi(i) = \text{const} \times e^{-1\cdot53 \log_e i - 0\cdot072 (\log_e i)^2} \qquad (23)$$

This may be compared with Seeliger's result (22).

Again, if we start from any number on the left side of Table 35, and move diagonally upwards to the right, the successive numbers will all refer to stars of the same

absolute magnitude. For example, starting from 17·2, we have the numbers

17·2	71·1	240	885	2770	7310

referring to an absolute magnitude $-4·4$ (strictly $-4·9$ to $-3·9$).

Now these are the numbers of stars in a series of spherical shells, the volumes of which form a geometrical progression—

1	4·0	15·8	63·1	251	1000

Whence by division the relative densities are

17·2	17·8	15·2	14·0	11·0	7·3

corresponding to distances of

25—40	40—63	63—100	100—158	158—251	251—398 parsecs.

According to our hypotheses, the change of star density with distance will be shown equally whatever absolute magnitude is selected; we shall thus obtain from Table 35 a number of determinations of the density law. The following numbers will illustrate the character of the density law deduced by Kapteyn.

Distance.	Star-density.
0	1·00
50 parsecs	0·99
135 ,,	0·86
213 ,,	0·67
540 ,,	0·30
850 ,,	0·15

It has been mentioned that in this second class of investigation, more data are employed than are necessary to give a solution. That is why we obtain from Table 35 a number of separate determinations of the luminosity and density laws instead of a single solution. Kapteyn has used the agreement of the separate determinations to defend the assumption that the absorption of light in space is not large. Schwarzschild has discussed the mutual consistency of the data analytically and has shown that the theoretical relations are well-satisfied.

Thus he finds that the theoretical value of the probable error of log $\pi/\bar{\pi}$ is 0·22, compared with 0·19 adopted by Kapteyn from the observations.

III. Investigations which Depend only on the Distribution of the Proper Motions of Stars.

Another class of statistical investigations depends entirely on the proper motions. In this it is convenient to introduce a new definition of the density law, viz., the number of stars *brighter than a limiting apparent magnitude* per unit volume of space at different distances from us. This involves a combination of the old density and luminosity laws, and in itself gives us no definite information as to the structure of the system; but it is clearly a matter of great practical interest to know how the stars of our catalogues are distributed in regard to distance. The determination of the velocity law is also an object of these investigations.

Let

$g(u)\,\frac{du}{u}$ be the number of stars having linear motions between u and $u+du$,

$h(a)\,\frac{da}{a}$ the number having proper motions between a and $a+da$;

$f(r)\,\frac{dr}{r}$ the proportion of stars (of the catalogue) at a distance between r and $r+dr$ from the Sun.

Then, expressing that $h(a)\,\frac{da}{a}$ is made up of stars at all possible distances r, with linear motions u between ra and $r(a+da)$.

$$h(a)\,\frac{da}{a} = \int_0^\infty \frac{f(r)\,dr}{r}\,\frac{g(ra)}{ra}\,r\,da.$$

Hence

$$h(a) = \int_0^\infty f(r)\,g(ra)\,\frac{dr}{r} \quad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

Or writing

$$r = e^\lambda \qquad a = e^\mu \qquad u = e^\gamma$$
$$f(e^\lambda) = \mathfrak{f}(\lambda) \qquad h(e^\mu) = \mathfrak{h}(\mu) \qquad g(e^\gamma) = \mathfrak{g}(\gamma).$$

Equation (24) becomes

$$\mathfrak{h}(\mu) = \int_{-\infty}^{\infty} \mathfrak{f}(\lambda)\, \mathfrak{g}(\lambda+\mu)\, d\lambda \qquad (25)$$

This is of the same form as (4), and the solution is accordingly

$$H(q) = 2\pi\, F(-q)\, G(q) \qquad (26)$$

where F, G, H are the Fourier integrals corresponding to $\mathfrak{f}$, $\mathfrak{g}$, $\mathfrak{h}$.

$$F(q) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathfrak{f}(\lambda)\, e^{-\iota q\lambda}\, d\lambda \qquad (27)$$

and

$$\mathfrak{f}(\lambda) = \int_{-\infty}^{\infty} F(q)\, e^{\iota q\lambda}\, dq \qquad (28)$$

To obtain a starting point it is assumed that in the direction at right angles to the vertex of star-streaming the linear motions are distributed according to the error-law $e^{-h^2(u-V)^2}du$, where V is the component of the solar motion in that direction. The evidence for this has been discussed in Chapter VII. There is the advantage that both the two-drift and ellipsoidal theories agree on this point, so that it is a fair assumption in any comparison of the two theories. Confining attention to the stars moving in this direction, $\mathfrak{g}$ can be found, and hence G is determined by periodogram analysis. For the same direction, H is determined by an analysis of the observed motions, and then F is given by (26). Having once found F (which determines the density law $\mathfrak{f}$), we may use its value in (26) to determine G for other directions, and $\mathfrak{g}$ is determined from G by another periodogram analysis (equation (28)).

Thus from the one assumption that the proper motions at right angles to the direction of star-streaming are distributed according to Maxwell's law, it is possible to determine the complete velocity-law. The method is a little difficult to apply, not only on account of the long

numerical computations, but also because the formulæ, which express *statistical* truths, cannot be exactly satisfied by observations depending on a limited sample of stars. The formal solution, which is unpleasantly conscientious, knows only one way out of such a difficulty; it diverges. We therefore have to smooth the observed data beforehand so as to make sure that the task is not an impossible one; and even then, if a trifling irregularity is left in, it

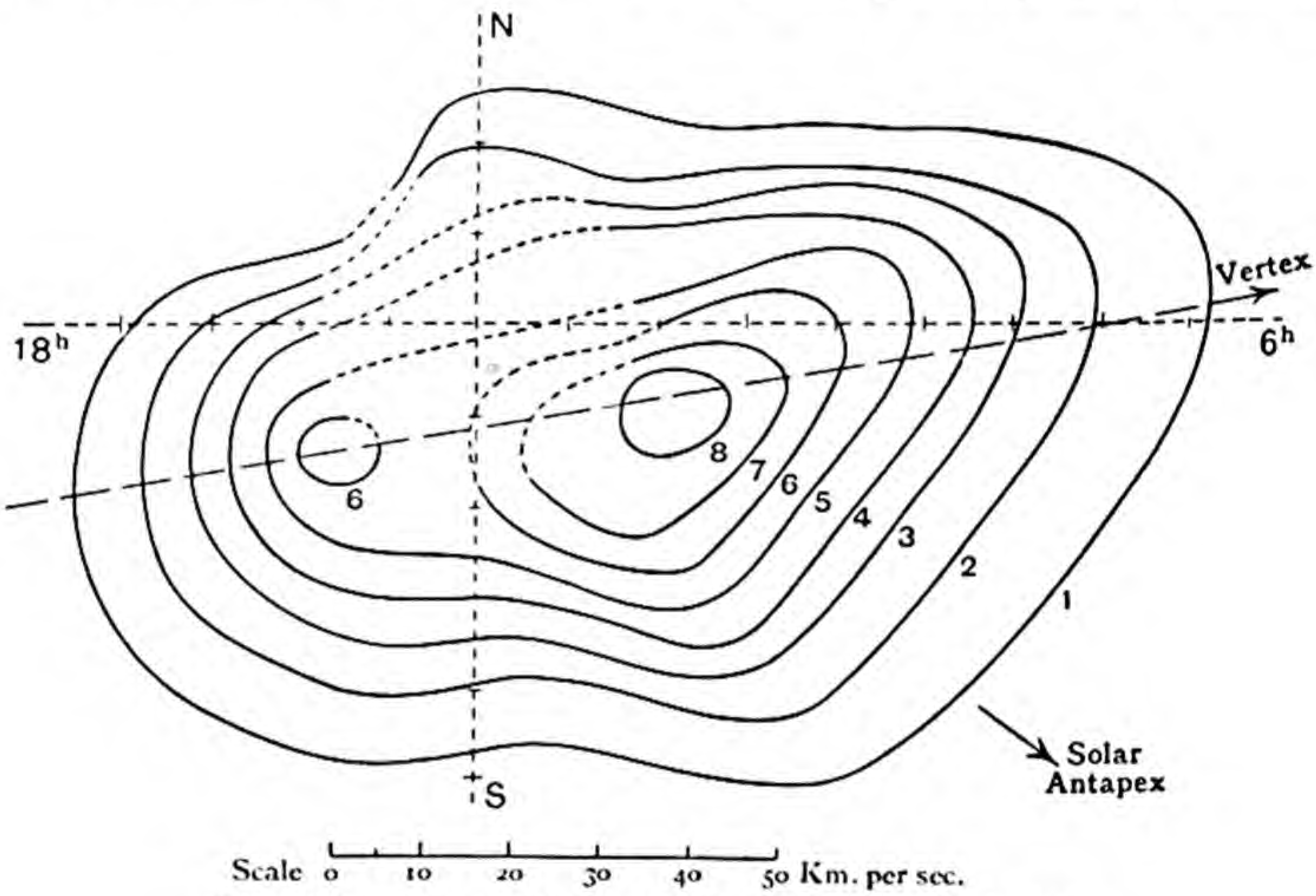

FIG. 21.—Curves of Equal Frequency of Velocity.

may cause the solution to make the most astounding oscillations in the attempt to follow it exactly.

This method has been applied to 1129 stars, the proper motions being taken from Boss's Preliminary General Catalogue. The stars are in two opposite areas of the sky with centres at the two Equinoctial points. As these centres are nearly 90° from the vertex and from the solar apex, the star-streaming and solar motion are shown in this region without foreshortening.

The resulting distribution of the linear velocities is shown in Fig. 21, which shows the curves of equal frequency. Setting the number of stars with component

velocities between x and $x+dx$, y and $y+dy$ proportional to $\psi(x,y)\,dx\,dy$, the curves $\psi(x,y)=1, 2, 3$, etc., have been traced. In the neighbourhood of the origin the present method of calculating ψ becomes indeterminate; accordingly within a certain distance of the origin the equi-frequental lines have not been calculated, but the diagram has been completed by the dotted lines, for it is sufficiently obvious how the joining up must take place. With the origin of co-ordinates shown in the figure the velocities are referred to the Sun; they may be referred to any other standard by merely changing the origin.

It will be seen that the diagram lends some support to the hypothesis of two distinct drifts as opposed to the unitary ellipsoidal hypothesis. It is not clear how far this indication of a division is to be trusted. I do not think that any modification of the analytical method or of the original assumptions would make any difference in the result; but the data may be scarcely sufficient.

As $F(q)$ is determined in the course of the analysis we may also derive the distribution of the stars in regard to distance. The results are as follows:—

TABLE 36.

Distribution of the Stars of Boss's Catalogue (for a mean Galactic Latitude 63°).

Distance.	Parallax.	Percentage of Stars.
Parsecs.	″ ″	
10·0— 12·5	0·10 —0·08	1·0
12·5— 16·7	0·08 —0·06	1·5
16·7— 25·0	0·06 —0·04	2·9
25·0— 50·0	0·04 —0·02	9·5
50·0— 66·7	0·02 —0·015	8·6
66·7— 100	0·015—0·010	17·5
100 — 125	0·010—0·008	11·7
125 — 167	0·008—0·006	14·9
167 — 250	0·006—0·004	16·6
250 — 500	0·004—0·002	10·0
500 —1000	0·002—0·001	2·4

No less than 70 per cent. of these stars are between 50 and 250 parsecs distant from us. It is an even chance that a star's parallax is contained within the limits $0''\cdot012$ and $0''\cdot004$; in other words, *if we know no more of a star than that it is brighter than* $6^{m}\cdot5$, *we may set down its parallax as* $0''\cdot008 \pm 0''\cdot004$.*

In the foregoing investigation special attention was devoted to the calculation of the velocity-distribution with as little assumption as possible, and the distance-distribution was found rather as a bye-product. We shall now consider researches in which the leading object is to find the distance-distribution.

Consider as usual the stars in an area of the sky sufficiently small to be treated as plane.

Let the proper motions be resolved into components along the direction towards the vertex and at right angles to it (in the plane of the sky). Denote the latter components by η.†

Let

$-V$ = Component of the solar motion (in linear measure) in the η direction.

$h(\eta)\,d\eta$ = Number of stars whose component η of proper motion lies between η and $\eta+d\eta$.

$g(v)\,dv$ = Proportion of stars whose corresponding component of linear motion lies between v and $v+dv$.

$f(r)\,dr$ = Number of stars (in the region) at a distance r to $r+dr$; so that the density of the stars brighter than the limiting magnitude at a distance r from the Sun is proportional to $r^{-2}f(r)$.

Let further

$H(\eta) = \int_0^\eta h(\eta)\,d\eta$ = Number of stars whose component proper motion is between 0 and η.

$F(r) = \int_0^r f(r)\,dr$ = Number of stars whose distance is less than r.

* This remark is strictly true according to the ordinary definition of probable error, but in a skew distribution of frequency such as this the probable error has not the properties we usually associate with it.

† The notation of the previous investigation is not conveniently applicable to this. We make an entirely fresh start.

The units of r, v, and η are made to correspond, so that

$$v = r\eta.$$

It is not difficult to see that the number of proper motions greater than η is to be found by multiplying the number of stars nearer than r by the proportion with linear velocities between ηr and $\eta(r+dr)$, for successive steps of r. Thus if N_1 and N_2 be the total numbers of stars with positive and negative components η, we shall have

$$N_1 - H(\eta) = \int_0^\infty F(r)\, g(r\eta)\, \eta\, dr,$$

and

$$N_2 - H(-\eta) = \int_0^\infty F(r)\, g(-r\eta)\, \eta\, dr,$$

so that

$$N_1 + N_2 - \{H(\eta) + H(-\eta)\} = \int_{-\infty}^\infty F(r)\, g(r\eta)\, \eta\, dr \quad . \; . \; . \quad (29)$$

provided that we choose an even function to represent $F(r)$.

We assume that the η components are distributed according to the Maxwellian Law, so that

$$g(v) = \frac{h}{\sqrt{\pi}} e^{-h^2(v-V)^2} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (30)$$

Consider now the special form :

$$\left.\begin{aligned} f(r) &= 2h^2k^2r\, e^{-h^2k^2r^2} \\ F(r) &= 1 - e^{-h^2k^2r^2} \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (31)$$

where k is a disposable constant.

Substituting in (29) and setting

$$\eta = nk,$$
$$\tau = hV,$$

the equation reduces to

$$\begin{aligned} H(\eta) + H(-\eta) &= \frac{n}{\sqrt{(1+n^2)}} e^{\tau^2/(1+n^2)} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (32) \\ &= R(n), \text{ say.} \end{aligned}$$

The quantity τ can be found from the ratio of the number of negative proper motions η to the whole number. This ratio is, in fact,

$$\frac{1}{\sqrt{\pi}} \int_\tau^\infty e^{-x^2} dx.$$

Thus the function $R(n)$ for the particular region can be tabulated. If then we can find a quantity k such that the proportion of stars having component proper motions numerically less than nk is represented by $R(n)$, then the distance distribution is given by (31).

The form of f here discussed has been selected (by F. W. Dyson and the writer independently) from other forms lending themselves to mathematical treatment, as corresponding most nearly to the observed motions. We may make a second approximation by superposing two such functions with different values of k.

Turning now to the results, we give first the distribution of stars brighter than $5^{m}\cdot8$ (Harvard Scale) for Types A and K separately. It has been pointed out previously that the assumed law (30) is only valid when stars of homogeneous type are discussed. Two regions were investigated, one with centre at the equinoxes (Gal. Lat. 63°), the other with centre at the poles (Gal. Lat. 27°); the former was much the more favourable for the purpose.

TABLE 37.

Distribution of Stars brighter than $5^{m}\cdot8$.

Limits of Distance.	Type K.		Type A.	
Parsecs.	G. Lat. 63°.	G. Lat. 27°.	G. Lat. 63°.	G. Lat. 27°.*
0— 25	11·7	9·2	7·9	7·5
25— 50	30·7	24·1	18·2	21·4
50— 75	35·3	31·5	18·8	32·7
75—100	31·4	31·8	17·6	38·5
100—125	24·8	31·0	16·7	40·9
125—150	19·0	28·7	16·3	39·0
150—175	14·5	26·5	16·0	34·5
175—200	11·3	23·3	15·0	27·9
200—225	8·2	19·5	13·2	21·3
225—250	5·7	15·6	11·2	15·3
>250	8·4	36·3	32·6	25·0

* North Polar area only. There appeared to be some abnormality near the South Pole.

Each region occupies 0·235 of the whole sky, and the figures give the actual number of stars.

The results for Type K, which appear to be the more trustworthy, show very well the falling off in numbers at great distances in high galactic latitudes as compared with low. The thinning out begins to be perceptible beyond 100 parsecs, and at 250 parsecs the density is only about one-quarter of its value in the lower latitude. The decrease is more rapid than would be expected from Seeliger's figures. The results for Type A are not so easy to accept. I am inclined to believe, however, that there must be a real clustering of Type A stars in the low region at a distance of about 100 parsecs, as the figures suggest. Perhaps the cluster has special properties as regards motion, which will upset the details of our table. In the region of high latitude (for which there seems no good reason to doubt the results) the stars appear much more spread out than Type K, as would be expected if Type A is rarer in space but of greater average luminosity. It seems possible that the unexpectedly low average distance of Type A (*see* p. 168) may be due to the presence of extensive clusters of these stars near the Sun. It was remarked that the next preceding type, B, shows a great tendency to form moving clusters, and possibly the same phenomenon in a vaguer form may remain in Type A.

For the distribution of the fainter stars the proper motions of 3,735 stars within 9° of the North Pole contained in Carrington's Circumpolar Catalogue are available. The limiting magnitude, which is the same as that of the Bonn Durchmusterung, is about $10^{m}\cdot 3$. These stars have been investigated by F. W. Dyson by a method similar in principle to that we have just described. Dyson used two forms of $f(r)$, viz.

$$(a)\quad f(r) = r\, e^{-h^2k^2r^2}$$

$$(b)\quad f(r) = r^{0\cdot 8}\, e^{-h^2k^2r^2}.$$

The differences in the resulting distributions are small; the former probably gives the closer representation of the nearer stars, and the latter of the more distant stars. Both results are given in Table 38.

TABLE 38.

Distribution of Stars brighter than $10^{m}\cdot 3$ *(for Galactic Latitude* 27°).

Limits of Distance.	Percentage of Stars.	
Parsecs.	Formula (*a*).	Formula (*b*).
0— 40	0·9	1·0
40— 100	5·0	4·7
100— 200	15·1	13·2
200— 400	40·1	35·2
400— 667	31·5	32·9
667—1000	7·1	11·9
>1000	0·3	1·1

As might be expected, these stars are considerably more distant than those of Boss's Catalogue (cf. Table 37). For example, only 5·7 per cent. are within 100 parsecs of the Sun, as compared with more than 40 per cent. of the Boss stars.

It may also be noted that 70 per cent. of the whole number have parallaxes between 0″·005 and 0″·0015; or, in terms of distance, 70 per cent. are distant between 200 and 650 parsecs.

It is believed that in low galactic latitudes the actual density of the stars in space is fairly constant up to a very considerable distance from the Sun. On this hypothesis Tables 37 and 38 enable us to determine the luminosity law; for the limiting magnitude of the stars considered corresponds to a limiting absolute luminosity which decreases as the square of the distance. If in the foregoing investigations—

$f(r)\,dr$ is the number of stars between the limits of distance r to $r+dr$ in an area of the sky ω,

m is the limiting magnitude of the catalogue,

$0^m \cdot 5$ is the Sun's stellar magnitude if removed to a distance of 1 parsec,

then the limiting luminosity for stars at a distance r is

$$(2\cdot512)^{0\cdot5-m} \times r^2,$$

and the number of stars per cubic parsec with luminosities greater than this is

$$\frac{f(r)}{\omega r^2}.$$

In Table 39 are given the luminosity laws derived from the Boss proper motions (Type K only) and the Carrington proper motions (all types). Owing to the difference in brightness of the stars used, the two investigations apply mainly to different parts of the luminosity curve.

TABLE 39.

Number of Stars in $4\cdot2\times10^6$ *units of space—equal to a sphere of radius 100 parsecs.*

Boss Stars (Type K only).		Carrington Stars (All Types)	
Luminosity (Sun = 1).	Number of Stars.	Luminosity (Sun = 1).	Number of Stars.
>500	10·8	>200	0
400 to 500	7·3	100 to 200	24
300 ,, 400	12·9	50 ,, 100	316
200 ,, 300	26·3	25 ,, 50	1,190
150 ,, 200	25·0	10 ,, 25	3,310
100 ,, 150	51·9	1 ,, 10	18,360
50 ,, 100	139·0	0·1 ,, 1	70,100
25 ,, 50	251·9		
10 ,, 25	500		
1 ,, 10	2,700		
Total brighter than the Sun	3,725	Total brighter than the Sun	23,200

The agreement is not particularly good. We should expect the numbers for Type K only to be considerably

less than for all the types taken together, so that for luminosities less than 100 there is no noteworthy discordance. For the Carrington stars, the number with luminosity greater than 100 depends on stars distant more than 900 parsecs; as these are not more than 2 per cent. of the whole, it is doubtful if the formulæ ought to be pressed so far. Also it may well be that at so great a distance there is a real diminution of the density of distribution of the stars in space.

It is interesting to note that more than 95 per cent. of the stars of Carrington (or of the Bonn Durchmusterung), and nearly 99 per cent. of the stars brighter than $5^m{\cdot}8$, are more luminous than the Sun. For a star of brightness equal to the Sun to appear as bright as $10^m{\cdot}3$ or $5^m{\cdot}8$ it must be distant less than 91 or 11 parsecs, respectively. The percentages given are the proportions outside these limits.

Adopting the law of distance-distribution for stars down to a limiting magnitude

$$f(r) = r\,e^{-h^2k^2r^2},$$

we may investigate the velocity-law in the direction of star-streaming. This has been done by Dyson for the Carrington proper motions. He found the best general agreement was given by Schwarzschild's ellipsoidal hypothesis rather than by the two-drift hypothesis. The ratio of the two axes of the velocity ellipse was found to be 0·60, in close agreement with the investigations of much brighter stars by the methods of Chapter VII. The two-drift hypothesis does not represent the small proper motions at all closely, though the existence of two maxima in the observed distribution of motions lends support to the view that there is some sort of separation of the two streams. The disagreement of the small proper motions is perhaps not surprising, for it is just this defect that the third Drift O was introduced to remedy; but it is certainly to the credit of the ellipsoidal hypothesis that

with so few constants it can lead to a good agreement with the small and large motions simultaneously.

The main points of discrimination between the two-drift and ellipsoidal hypotheses appear to be

(1) The skewness of the velocity-distribution.
(2) The spread of the distribution.
(3) The existence (or not) of two maxima.

In the first point the advantage is admittedly with the two-drift hypothesis, and it seems equally certain that the ellipsoidal hypothesis is better fitted to the second. With regard to (3) the evidence is rather in favour of the two drifts. According to the relative importance attached to these three criteria, different views of the merits of the two methods of approximation will prevail.

One point of great practical importance must be attended to in investigations of the kind we have been considering (Section III.). The distribution of the proper motions is generally appreciably altered by the presence of accidental error in the observations. Usually the probable error of the observations is known, and in this case we can, from a table of the number of observed proper motions between given limits, form a revised table giving the true number, *i.e.*, we can correct the observed values of $h(\eta)$ for the effects of the known accidental error.[6] Unless the accidental error is large, the correction to be applied to each number of the observed table is

$$-\left(\frac{1\cdot046 \times \text{probable error}}{\text{tabular interval}}\right)^2 \times \text{tabular second difference.}$$

The full formula, applicable to any kind of statistics, is—

$$v(m) = \exp\left(-\frac{1}{4h^2}\frac{d^2}{dm^2}\right) u(m),$$

where $v(m)$ and $u(m)$ are the true and observed frequency functions, and $0\cdot477/h$ is the probable error of the observations of m.

General Conclusion.

The investigations described in this chapter suffer from all the imperfections inseparable from pioneer work. The numerical results in Sections I. and II. would, I believe, be modified in an important degree, by a modern determination of the mean parallaxes of stars of different magnitudes; those of Section III. suffer from scantiness of data, insufficient range, and the absence of an effective check on the main assumptions. Yet I do not think it can be doubted that these general statistical researches have already greatly advanced our knowledge of the distribution and luminosities of the stars. If the approximation is not yet a close one, our present vague knowledge is very different from the complete ignorance from which we started. But the main interest of this chapter lies in the hope for the future.

It is of special importance to note that there exist two entirely independent methods of determining the distribution in distance of stars brighter than a limiting magnitude, the one resting on magnitude-counts and mean parallactic motions (Section I.),* and the other on the distribution of the individual proper motions (Section III.). There is nothing in common between the data used for these two methods; and the one is a complete check on the other. When the time arrives that this check is satisfied, and that results obtained along one line of investigation are in full agreement with those obtained along an independent line, the results of these methods of research will have been placed on a firm basis. Meanwhile, the conclusion that such a check is possible may be regarded as one of the most useful results of these preliminary discussions.

* The method of Section I. leads, as we have seen, to a complete solution for the luminosity-law and density-law. The distribution in distance of stars brighter than a limited magnitude is easily derived from these.

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CHAPTER XI

THE MILKY WAY, STAR-CLUSTERS, AND NEBULÆ

At the beginning of the preceding chapter we emphasised the fact that the statistical investigations referred to an idealised sidereal system, which retained some of the more important properties of the actual universe, but neglected many of the details of the distribution. If an impression has been given that the spheroidal distribution of stars with density diminishing outwards is a complete and sufficient model, a glance at Plate I (*Frontispiece*), which is a photograph of the region of the Milky Way in the neighbourhood of Sagittarius, may serve to correct this. In the Milky Way there are unmistakable signs of clustering and irregularities of density on a large scale. The great star-clouds and deep rifts are features in marked contrast with the phenomena of distribution that we have hitherto considered, and no elaboration of the theory of a disc-shaped or spheroidal system will suffice to explain them. This does not affect our conclusions as to the shape of what we have called the inner stellar system; a general concentration of stars to the galactic plane is manifested quite apart from the great clusters of the Milky Way itself. It would certainly be desirable in discussing problems such as those of the last chapter to ignore, or at least treat separately, the parts of the sky through which the Milky Way itself passes, for our idealised system evidently becomes inadequate here.

The view then is taken that there is first an inner stellar system consisting of a flattened distribution of stars of density more or less uniform at the centre and diminishing outwards; and secondly a mass of star-clouds, arranged round it and in its plane, which make up the Milky Way (*see* Fig. 1, p. 31). It is to the inner system that our knowledge of stellar motions and luminosity relates. Whether the outer clouds are continuous with the inner system or whether they are isolated, is a question at present without answer.

It is usual to consider the system in the midst of which the Sun lies as the principal system, the clusters of the Milky Way being a kind of appanage. An alternative view makes no such distinction, but contemplates a number of star-clouds scattered irregularly in the one fundamental plane, our own system being one of them. There are certain advantages in the latter view, especially as the two star-streams could then be accounted for by two of these star-clouds meeting and passing through one another. By a natural reaction from the geocentric views of the Middle Ages, we are averse to placing the earth at the hub of the stellar universe, even though that distinction is shared by thousands of other bodies. But it is doubtful if there is really any close resemblance between the Milky Way aggregations and that which surrounds the Sun. We do not recognise in them the oblate form flattened in the fundamental plane, which is so significant a feature of the solar star-cloud. They seem to be of a more irregular character and for that reason we prefer to adhere to a theory which regards them as subsidiary.

The great mass of the stars shown in photographs of the Milky Way are very faint. We have no knowledge of their motions or their spectra, and even now there is but little accurate information as to their magnitudes and numbers. It is an important question whether some of the bright stars, which are seen in the same region as these

star-clouds, are actually in the clouds or only projected against them. The investigations of this point are rather contradictory; but on the whole it seems probable that some stars of the sixth magnitude are actually situated in the Milky Way clusters. Certainly by the ninth magnitude we have begun to penetrate into the true galaxy, and the twelfth or thirteenth magnitude takes us right into the heart of the aggregations. Simon Newcomb[1] attacked the problem by comparing the density of the lucid stars, where the Milky Way background was respectively bright and faint; he found that bright stars were most numerous where the background was bright.

TABLE 40.

Relation of Lucid Stars to the Milky Way (Newcomb).

	N. Hemisphere (Limiting Mag. 6·3).	S. Hemisphere (Limiting Mag. 7·0).
Mean star-density of whole hemisphere	19·0	32·7
Star-density: darker galactic regions.	20·4	33·8
,, brighter ,, ,,	32·9	79·4

The star-density is given per 100 square degrees. The greater value in the southern hemisphere is due to the lower limit of magnitude in the catalogue which was used.

The condensation of the stars to the brighter regions is very marked; but some caution is needed in interpreting this result. There is no doubt that many of the dark patches in the Milky Way are due to the absorption of light by tracts of nebulous matter. To perform their work of absorption these tracts must lie on the nearer side of the Milky Way aggregations. Since Table 40 shows that the star-density in the darker regions is barely greater than the mean for the hemisphere, and therefore less than in a zone just outside the Milky Way, the dark matter must be at

least partly within the oblate inner system. Newcomb's result therefore teaches us that some of the sixth and seventh magnitude stars lie in and beyond the dark clouds; but it is not conclusive that any of them lie in the bright aggregations of the Milky Way. To prove the latter result we should have to show that the star-density in the bright regions is greater than could reasonably be attributed to the oblate shape of the inner system; the figures suggest that this is so, but there is room for doubt.

A similar discussion of fainter stars has been made by C. Easton,[2] who considered especially the part of the Milky Way in Cygnus and Aquila, where there is a wide range in the intensity of the light. A selection from his results is given in Table 41.

TABLE 41.

Relation of Stars to the Milky Way (Easton).

	Argelander Durchmusterung (Mag. 0—10).	Wolf Photographs (Mag. 0—11).	W. Herschel Star-gauges (Mag. 0—14).
Star-density—			
Darkest patches	23	72	405
Intermediate ,,	33	134	4114
Brightest ,,	48	217	6920

The density is given per square degree.

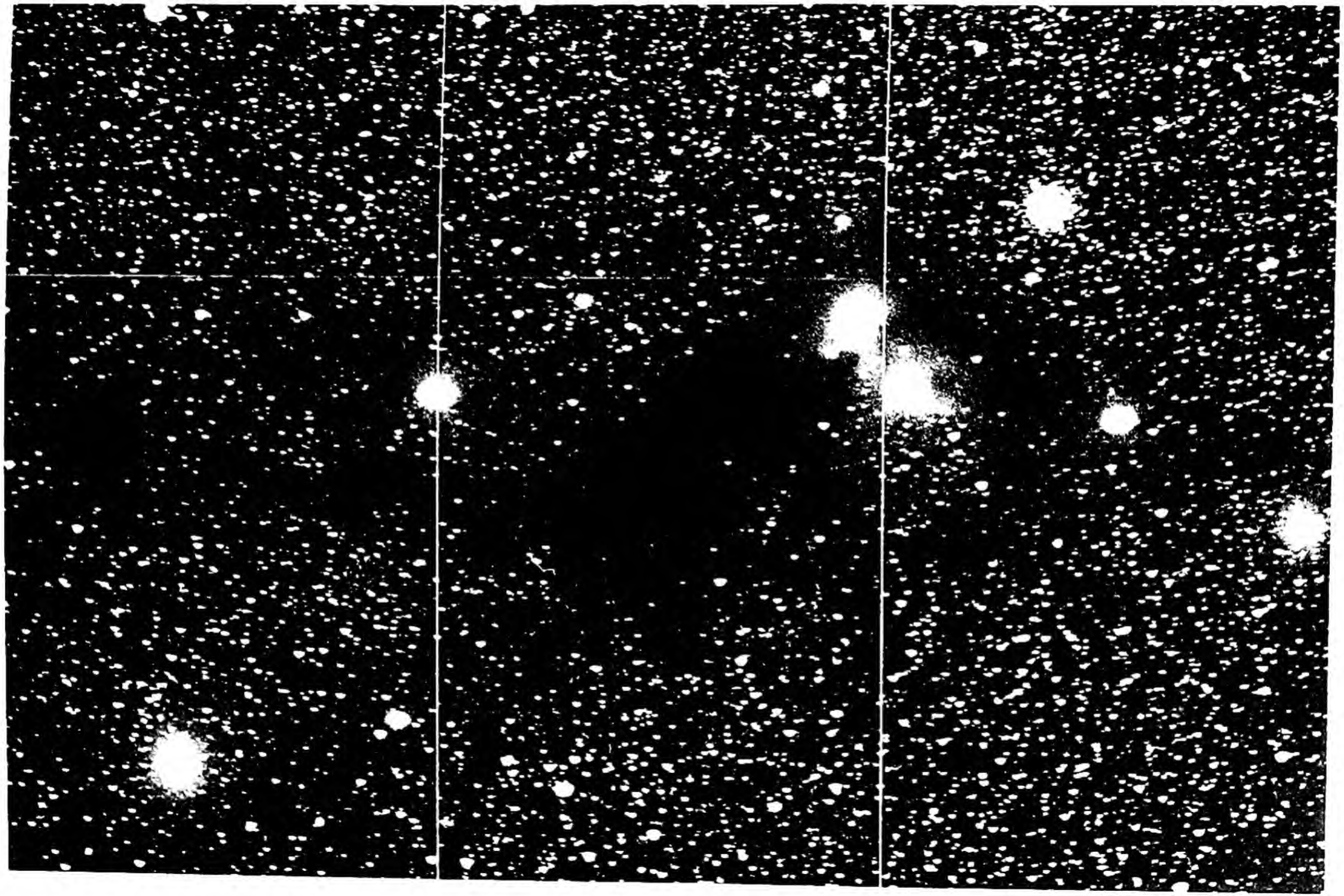
The numbers show that as we proceed to fainter stars a rapidly increasing proportion is associated with the true Milky Way aggregations, and by the time the fourteenth magnitude is reached, an overwhelming proportion is found to belong to them. But Easton's results and Newcomb's are not quite in agreement. The relative superiority of the bright patches found by Easton for 10^m is barely greater than that found by Newcomb for 6^m–7^m. Indeed by extrapolation of Easton's figures we should conclude that the stars brighter than 7^m were not notice-

ably associated with the Milky Way background. If we could suppose that the Cygnus-Aquila region is more distant than the average, the difference between the two results would be explained; but Easton has given reasons for believing that this region is nearer than the average.

In all the counts discussed by Easton, the star-density of the bright patches is so notably superior to the density just outside the Milky Way that we must conclude that the excess is actually due to the star-clouds. This method of considering the problem is so straightforward that it seems impossible to doubt the conclusions. The results should be independent of systematic errors in the limits of the counts, for the bright and dark regions adjoin one another and are irregularly intermixed. If there is any tendency to make the counts less complete in the rich regions or where the background is bright, this only means that the difference is really more accentuated than is shown in the Table.

If it is established that the Milky Way aggregations include a fair number of stars that appear to us of the ninth magnitude, it is possible to form some conception of their distance. A star of luminosity 10,000 times that of the Sun would appear of the ninth magnitude at a distance of 5,000 parsecs. This may be taken as an upper limit for the distance of the nearer parts of the Milky Way. If, following Newcomb, we admit the presence of sixth magnitude stars in the aggregations, the limit is reduced to 1200 parsecs. In any case the so-called "holes" in the Milky Way (dark nebulæ) would seem to be in some cases within this latter distance. There is no reason to believe that all parts of the galaxy are at the same distance, and certain appearances suggest that there may be two or more branches lying behind one another in some parts of the sky. The relative brightness of the different portions gives no clue to the distance; for the apparent brightness (per unit angular area) of a

Plate 2



NEBULOUS DARK SPACE IN [illegible]

cluster of stars is independent of its distance.* The differences in intensity must, therefore, be due either to a greater depth in the line of sight or to a closer concentration of the stars. (If there is appreciable general absorption of light in space, this statement would have to be modified.) Further, as it is necessary to suppose that the whole structure is quite irregular, it would be very unsafe to assume that the angular width of the different portions gives any measure of the distance.

It is well known that the large irregular nebulæ are found principally in the Milky Way, differing in this respect from the great majority of the compact nebulæ. In the "irregular" class we include not only the more intense patches, such as the Omega, Keyhole, and Trifid nebulæ, etc., but the extended nebulous backgrounds, such as those photographed by E. E. Barnard in Taurus, Scorpius, and other constellations. Of the same nature are many of the dark spaces in the Milky Way, where the light of the stars behind is cut off by nebulous regions which give little or no visible light of their own. These dark spaces are usually connected with diffused visible nebulosity, which often surrounds and is condensed about one or more bright stars. An excellent example is found in the constellation Corona Austrina (Plate 2), where there is a dark area containing very few stars, edged in some parts with visible nebulosity, which condenses into bright cusps round the bright stars. No one examining the photographs can doubt that the darkness is caused by the nebula the edge of which is visible. Visual observers have asserted that the region has a leaden or slightly tinted appearance as though a cloud were covering part of the field.[3] Another curious region is found in Sagittarius near the cluster Messier 22, where great curved lanes are, as it were, smudged out among the thickly scattered stars. The

* Nearness might decrease the background illumination somewhat, for a greater amount of the light would appear in the form of distinct stars.

impression is irresistible that the effect is produced by absorbing nebulosity. The obliterated regions often take the form of narrow lanes having a nebula at one end; a favourite example is Max Wolf's cave-forming nebula in Cygnus. The general tendency of the gaseous nebulæ to sweep clear a space among the stars was pointed out by Sir W. Herschel[4] and has long been a recognised phenomenon. It must not be assumed that all the rifts of the Milky Way are to be explained in this way; especially such regions as are poor only by contrast with the dense star-clouds, and are not darker than the normal sky. But there is abundant evidence that absorbing material creates many of the curious markings which appear.

These irregular nebulæ (luminous as well as dark) are found almost exclusively in the Milky Way. This may be due to the general oblate form of the stellar system or to an actual association of the nebulæ with the galactic aggregations of star-clouds. The latter explanation is most usually accepted and appears most probable. Something may be learned from photographs as to whether there is an association of structure between the nebulosity and the star aggregations; no doubt the general impression given is that the two are closely related, but it is difficult to find definite evidence. We cannot learn much from the dark rifts, for they may be attributed to material absorbing star-light and nebula-light alike; though it may be significant that they occur especially in the most brilliant star clouds. In certain cases, as, for instance, in Orion and Corona Austrina, we know that the stars are actually associated with and in the midst of the nebula; but it is not certain that these stars belong to the Milky Way. In view of their great brightness that is a daring assumption.

Besides the irregular nebulæ, many other classes of objects appear to be strongly condensed towards the

Milky Way. How far this is a reflection of the oblate shape of the stellar system, and how far it is a real association with the Milky Way formations, is a matter of doubt. We give in Table 42, a list (due to E. Hertzsprung[5]) of the celestial objects which are most strikingly concentrated, together with the pole of their plane of greatest concentration. The near agreement with the galactic pole is very remarkable.

TABLE 42.

Objects Showing Galactic Concentration (Hertzsprung).

Class.	Pole of Plane of Concentration. R.A.	Pole of Plane of Concentration. Dec.	No. of Objects.
Helium stars (Oe5 – B9)	182°·1	+27°·9	1402
Stars of Type N	194°·2	+27°·4	228
Wolf Rayet stars (Oa – Oe)	190°·7	+26°·9	87
Eclipsing Variables	188°·2	+25°·8	150
Cepheid Variables	195°·9	+26°·8	60
c and *ac* stars	189°·1	+26°·3	98
Gaseous Nebulæ	192°·7	+28°·1	130
Galactic Pole (Pickering)	190°·0	+28°·0	–

Of these bodies, the Wolf-Rayet stars present the most remarkable concentration. Of 91 known objects of this type, 70 are actually within the borders of the Milky Way, and the whole of the remaining 21 are in the Magellanic Clouds.[6] Excluding the latter, the mean distance from the central galactic circle is only 2°·6. It is perhaps of some significance that the great Andromeda Nebula shows a spectrum in which the chief Wolf-Rayet lines are believed to be present.

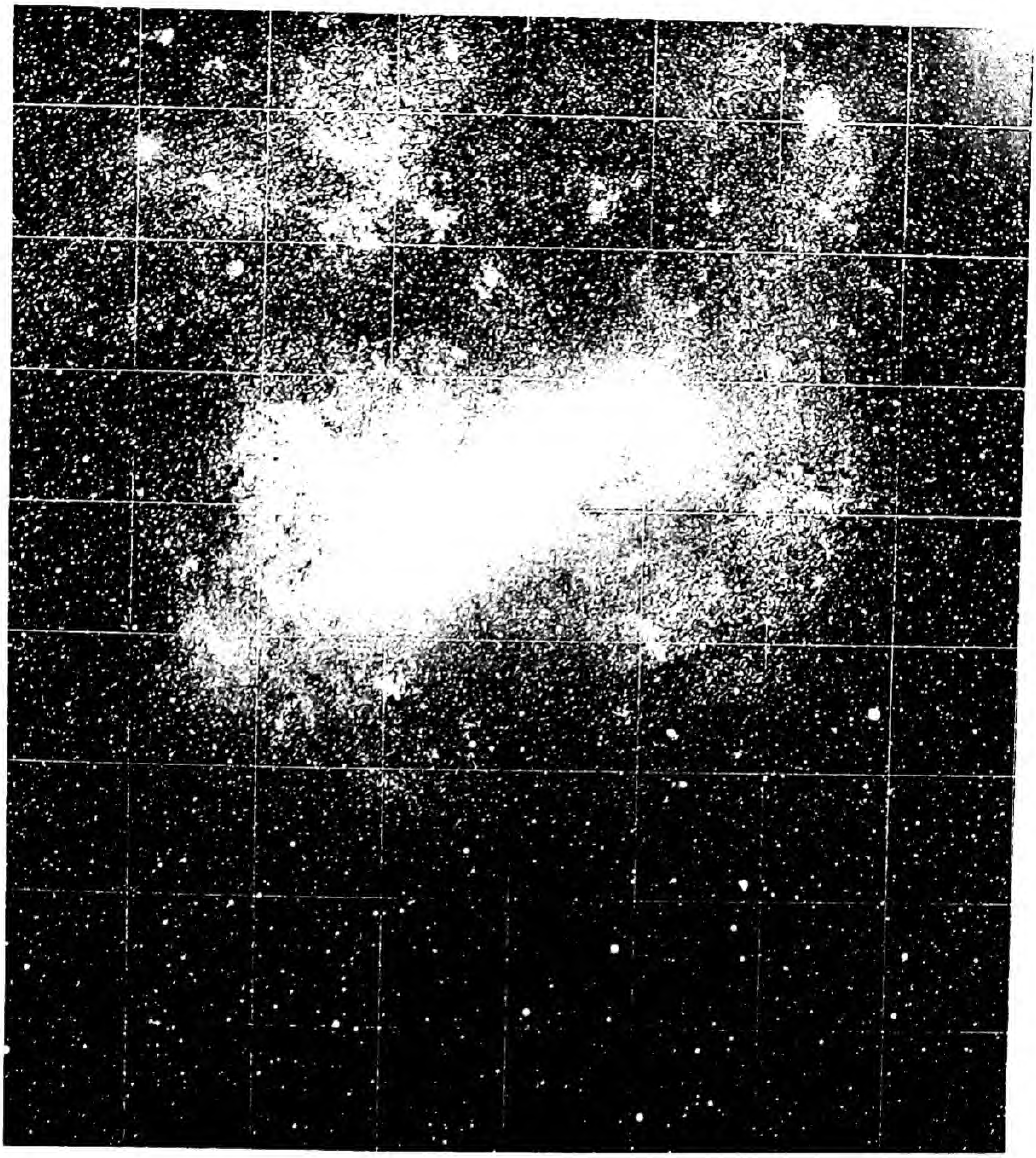
The Magellanic Clouds would at first sight seem to be isolated portions of the Milky Way clusters. They possess, however, some distinctive features; and their high galactic latitude is unfavourable to the view that there is any close

association. In the Greater Cloud (Plate 3) there are a large number of nebulous knots, which have generally been described as spiral (*i.e.*, non-gaseous) nebulæ. If this were really their nature it would form a remarkable distinction between the clouds and the Milky Way, for the spiral nebulæ avoid the latter. But according to A. R. Hinks the supposed nebulæ of the Magellanic Clouds are unlike anything found elsewhere, and have no resemblance to the true spiral nebulæ. Many of the principal nebulæ in the Cloud are undoubtedly gaseous.

By an ingenious argument E. Hertzsprung[7] has arrived at an estimate of the distance of the Lesser Magellanic Cloud, which is entitled to some confidence. It depends on the existence of a large number of variables of the δ Cephei type in that Cloud. Now there is reason to believe that the absolute magnitude of a Cepheid variable of given period is a fairly definite quantity; that, in fact, it can be predicted from the period with a mean uncertainty of only a quarter of a magnitude. This was shown by Miss Leavitt[8] who discussed the variables in the Lesser Cloud. Since these must be at nearly the same distance from the Sun, their apparent magnitudes will differ from their absolute magnitudes by a constant. She found that the magnitude and the logarithm of the period were connected by a linear relation, and that the average deviation of any individual from the general formula was $\pm 0^{m}{\cdot}27$. Now the mean distance of the brighter Cepheid Variables can be calculated from the parallactic motion in the usual way. It is then only necessary to multiply by the factor for the difference of magnitude between these and the Magellanic Variables and allow for the difference of period, if any, in order to obtain the distance of the latter. In this way the distance of the Lesser Magellanic Cloud is found to be 10,000 parsecs—the greatest distance we have yet had occasion to mention.

Passing from the large-scale aggregations of stars, we

Plate 3.



THE GREATER MAGELLANIC CLOUD.

must refer briefly to the star-clusters proper. There seems no reason to doubt that these are of the same nature as the moving clusters discussed in Chapter IV. In particular the Taurus-stream may be taken to be typical of the globular clusters, although it is not one of the richer specimens of the class. The distribution of these globular clusters in the sky is very remarkable; they are to be found almost exclusively in one hemisphere of the sky, the pole of which is in the galactic plane in galactic longitude 300°. This result (which is taken from A. R. Hinks's discussion [9]) is clearly of great significance; but it does not seem possible at present to attempt any explanation of it.

The chief characteristics, from our point of view, of the planetary nebulæ are their close condensation to the Milky Way and the large radial velocities of those that have been measured. Here again we are not able to do more than state the facts. I am not aware of any trustworthy measures of the proper motions of planetary nebulæ, and their size and distance are consequently a matter of extreme uncertainty; but their marked tendency to lie in the plane of the Milky Way shows that they must be placed somewhere within our own stellar system.

In Plate 4 is shown the Whirlpool Nebula in Canes Venatici, a fine example of the spiral nebulæ, which are among the most beautiful objects in the heavens. It is generally believed that the spirals predominate enormously over the other classes of nebulæ; and, as the whole number of nebulæ bright enough to be photographed has been estimated by E. A. Fath at 160,000, they must form a very numerous class of objects. They are seen by us at all inclinations, some, like the Whirlpool Nebula, in full front view, whilst others are edge-on to us and appear as little more than a narrow line. An example of the latter kind is also illustrated in Plate IV. In all cases, where it is possible to discriminate the details, the spiral is seen to be double-branched, the two arms leaving the nucleus at

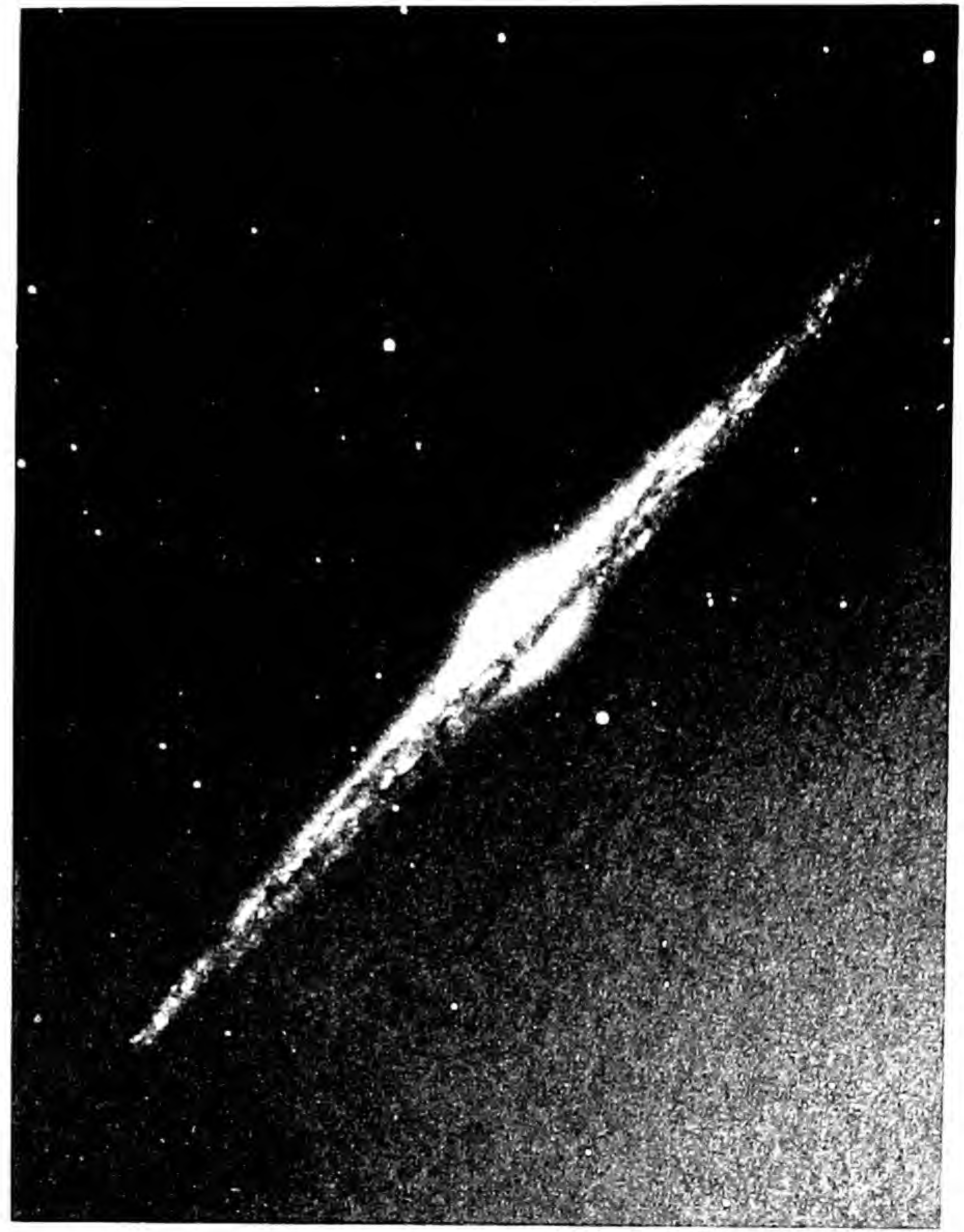
opposite points and coiling round in the same sense. From the researches of E. v. d. Pahlen [10] it appears that the standard form is a logarithmic spiral. The arms, however, often present irregularities, and numerous knots and variations of brightness occur. Unlike the planetary and extended nebulæ, the spectrum shows a strong continuous background ; bright lines and bands are believed to occur, at least in the Great Andromeda Nebula ; but they are of the character of those found in some of the early type stars, and are distinct from the emission lines of the gaseous nebulæ.

The distribution of spiral nebulæ presents one quite unique feature : they actually shun the galactic regions and preponderate in the neighbourhood of the galactic poles. The north galactic pole seems to be a more favoured region than the south. This avoidance of the Milky Way is not absolute ; but it represents a very strong tendency.

In the days before the spectroscope had enabled us to discriminate between different kinds of nebulæ, when all classes were looked upon as unresolved star-clusters, the opinion was widely held that these nebulæ were "island universes," separated from our own stellar system by a vast empty space. It is now known that the irregular gaseous nebulæ, such as that of Orion, are intimately related with the stars, and belong to our own system ; but the hypothesis has recently been revived so far as regards the spiral nebulæ. Although the same term "nebula" is used to denote the three classes—irregular, planetary, and spiral—we must not be misled into supposing that there is any close relation between these objects. All the evidence points to a wide distinction between them. We have no reason to believe that the arguments which convince us that the irregular and planetary nebulæ are within the stellar system apply to the spirals.

It must be admitted that direct evidence is entirely

Plate 4.



lacking as to whether these bodies are within or without the stellar system. Their distribution, so different from that of all other objects, may be considered to show that they have no unity with the rest; but there are other bodies, the stars of Type M for instance, which remain indifferent to galactic influence. Indeed, the mere fact that spiral nebulæ shun the galaxy may indicate that they are influenced by it. The alternative view is that, lying altogether outside our system, those that happen to be in low galactic latitudes are blotted out by great tracts of absorbing matter similar to those which form the dark spaces of the Milky Way.

If the spiral nebulæ are within the stellar system, we have no notion what their nature may be. That hypothesis leads to a full stop. It is true that according to one theory the solar system was evolved from a spiral nebula, but the term is here used only by a remote analogy with such objects as those depicted in the Plate. The spirals to which we are referring are, at any rate, too vast to give birth to a solar system, nor could they arise from the disruptive approach of two stars; we must at least credit them as capable of generating a star cluster.

If, however, it is assumed that these nebulæ are external to the stellar system, that they are in fact systems coequal with our own, we have at least an hypothesis which can be followed up, and may throw some light on the problems that have been before us. For this reason the "island universe" theory is much to be preferred as a working hypothesis; and its consequences are so helpful as to suggest a distinct probability of its truth.

If each spiral nebula is a stellar system, it follows that our own system is a spiral nebula. The oblate inner system of stars may be identified with the nucleus of the nebula, and the star clouds of the Milky Way form its spiral arms. There is one nebula seen edgewise (Plate IV) which makes an excellent model of our system, for the

oblate shape of the central portion is well-shown. From the distribution of the Wolf-Rayet stars and Cepheid Variables, believed to belong to the more distant parts of the system, we infer that the outer whorls of our system lie closely confined to the galactic plane; in the nebula these outer parts are seen in section as a narrow rectilinear streak. The photograph also shows a remarkable absorption of the light of the oblate nucleus, where it is crossed by the spiral arms. We have seen that the Milky Way contains dark patches of absorbing matter, which would give exactly this effect. Moreover, quite apart from the present theory, a spiral form of the Milky Way has been advocated. Probably there is more than one way of representing its structure by means of a double-armed spiral; but as an example the discussion of C. Easton [11] may be taken, which renders a very detailed explanation of the appearance. His scheme disagrees with our hypothesis in one respect, for he has placed the Sun well outside the central nucleus, which is situated according to his view in the rich galactic region of Cygnus.

The two arms of the spiral have an interesting meaning for us in connection with stellar movements. The form of the arms—a logarithmic spiral—has not as yet given any clue to the dynamics of spiral nebulæ. But though we do not understand the cause, we see that there is a widespread law compelling matter to flow in these forms.

It is clear too that either matter is flowing into the nucleus from the spiral branches or it is flowing out from the nucleus into the branches. It does not at present concern us in which direction the evolution is proceeding. In either case we have currents of matter in opposite directions at the points where the arms merge in the central aggregation. These currents must continue through the centre, for, as will be shown in the next chapter, the stars do not interfere with one another's paths. Here then we have an explanation of the prevalence of motions

to and fro in a particular straight line; it is the line from which the spiral branches start out. The two star-streams and the double-branched spirals arise from the same cause.

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CHAPTER XII

DYNAMICS OF THE STELLAR SYSTEM

During the time that the stars have been under observation their motion has been sensibly rectilinear and uniform. A reservation must be made in the case of binary stars, where the components revolve around one another; but from the present point of view pairs or multiple systems of this kind only count as single individuals. With this exception, we have no direct evidence that one star influences the motion of another; yet we cannot doubt that, in the vast period of time during which the stellar universe has been developing, the forces of gravitation must have played a part in shaping the motions that now exist. It may not be premature to consider the dynamical aspects of some of the discoveries of recent years.

The action of one star on another, even at the smallest normal stellar distance is exceedingly minute. The attraction of the Sun on α Centauri imparts to that star in the course of a year a velocity of one centimetre per hour. At this rate it would take 380,000,000 years to communicate a velocity of one kilometre per second. The period is not so excessive, compared with what we believe to be the life of a star, as to entitle us to despise such a force. But the two stars will not remain neighbours for more than a small fraction of that time. Although α Centauri is at present approaching, a separation will soon begin; in

150,000 years from now the distance will have doubled; and before the communicated velocity amounts to more than a fraction of a metre per second the star will have receded out of range of the Sun's attraction.

The case is different when we consider the general attraction of the whole stellar system on its members. Not only is the magnitude of the force somewhat greater, but the time through which its effects accumulate is far longer than in the case of one star acting on a temporary neighbour. This general attraction is quite sufficient to produce important effects on the stellar movements.

The field of force in which a star moves is due to a great number of point-centres—the stars. The distribution of the attracting matter is discontinuous. We therefore divide the force into two parts, (1) the attraction in an ideal continuous medium having the same average density, and the same large-scale variations of density as the stellar system, and (2) the force due to the accidental arrangement of the stars in the immediate neighbourhood. The same distinction occurs in the ordinary theory of attractions for points inside the gravitating matter. We call the first part the *general* or *central attraction* of the system; the term *central* is perhaps inaccurate for there is no true centre of attraction unless the system possesses spherical symmetry. The second part is of an accidental character and will act in different directions at different times; but it is not on that account to be dismissed without consideration.

The stars have often been compared to the molecules of a gas; and it has been proposed to apply the theory of gases to the stellar system.[1] The essential feature in gas-dynamics is the prominent part taken by the collisions of the molecules. Now it is clear that collisions of the stars, if they occur at all, must be exceedingly rare; and the effect would certainly not be the harmless rebound contemplated by the theory of gases. It is, however,

well-known that the precise mode of interaction during the encounter is of little importance, and all that is required in the theory is an interchange of momentum taking place between two individuals in their line of centres. In this generalised sense encounters are continually taking place; the passage of one star past another always involves some interchange of momentum. It remains to be examined whether this continuous transference can play the same part in stellar theory as the abrupt changes of momentum in the gas theory.

In the long run the same effect will be brought about. The ultimate state of a system of gravitating stars will be the same as that of a gas. The ultimate law of velocities will be the same as in a mass of non-radiating monatomic gas under its own attraction; and moreover there will be equipartition of energy between the stars of different masses, just as if they were atoms of different weights. We might even go further, and look forward to a still more "ultimate" state, in which the double stars behaved as diatomic molecules. But it is unnecessary to pursue these deductions, for they have no reference to anything in the present state of the stellar universe, or to any future near enough to interest us.

It was seen in Chapter IV. that the existence of the Moving Clusters shows plainly that the encounters have not as yet had any appreciable effect on the motions of the stars. Taking, for example, the Taurus Cluster, we have seen that it occupies a sphere of about 5 parsecs radius, which would in an ordinary way contain 30 stars. As it cannot be supposed that a special track has been cleared for the passage of the cluster, the stars that would naturally occupy the space must be there, permeating the cluster without belonging to it. In so far as they have any effect at all, the attractions of these interlopers must tend to break up and dissipate the cluster, by destroying the parallelism of the motions. As no such breaking up has

taken place, it may be inferred that the chance encounters have had no appreciable effect on the stellar velocities up to the present time. Many of the stars of the Taurus Cluster are in a mature stage of development, so that this inference may fairly be applied to the general mass of the stars.

A consideration of this question from the theoretical side is in entire accordance with this conclusion. We begin by considering the numerical amount of the deflection produced by an encounter in given circumstances.

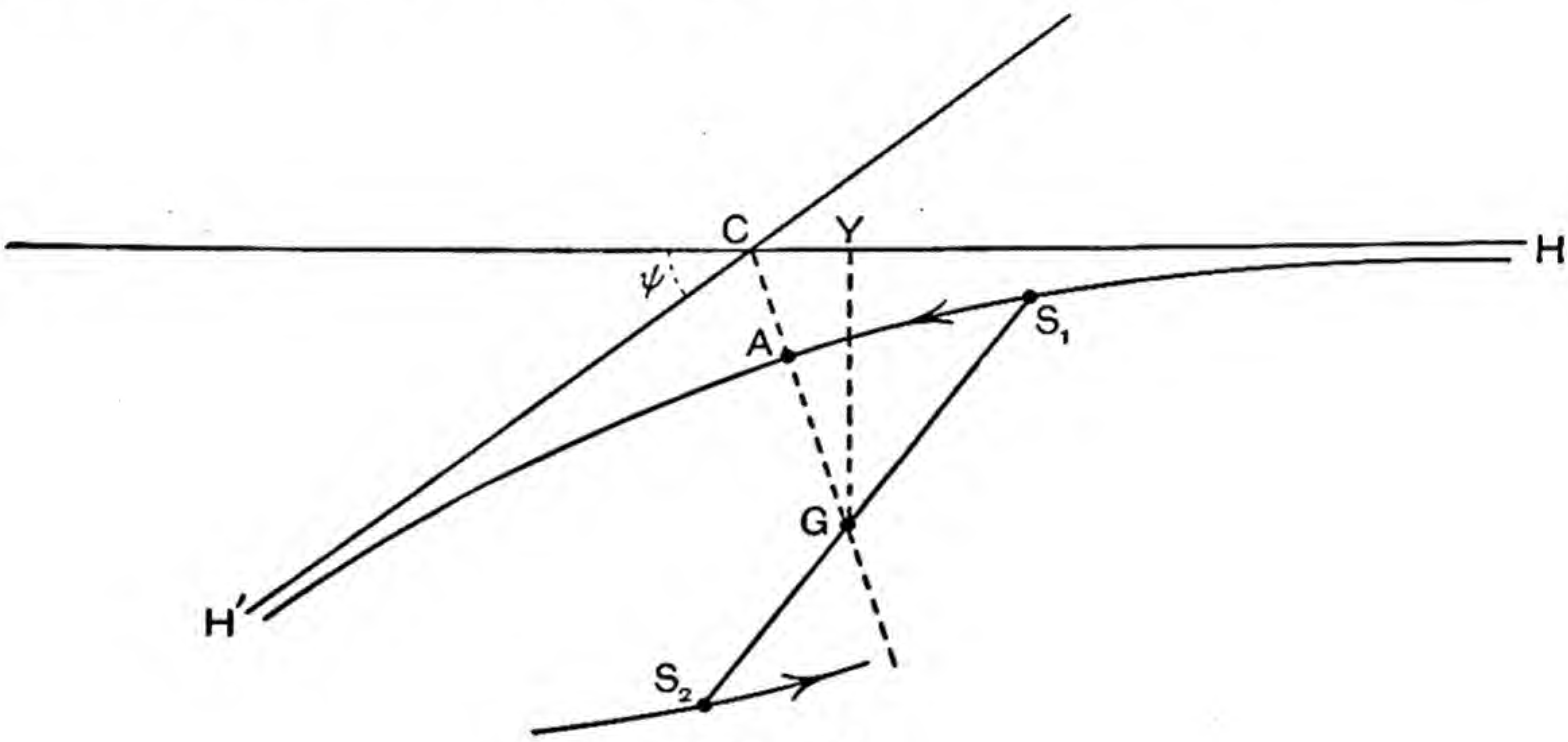

Fig. 22.

Let S_1 S_2 (Fig. 22) be two stars of masses m_1 and m_2 and G be their common centre of gravity.

Since

$$GS_1 = \frac{m_2}{m_1+m_2} \cdot S_2 S_1,$$

we may replace the star S_2, in considering its attraction on S_1, by a star of mass

$$m_2\left(\frac{m_2}{m_1+m_2}\right)^2 \text{ at } G.$$

Taking G to be at rest, let the star S_1 move along the hyperbolical path HAH' starting with an initial velocity V. Let CH, CH' be the asymptotes of the hyperbola.

Draw GY perpendicular to CH; then GY is equal to the transverse axis b of the hyperbola.

The usual equation $h^2 = \mu l$ gives in this case

$$(V.GY)^2 = \mu \frac{b^2}{a}.$$

Hence

$$a = \frac{\mu}{V^2}.$$

The deflection $\psi =$

$$180^\circ - HCH'$$

is given by

$$\tan \tfrac{1}{2}\psi = \frac{a}{b} = \frac{\mu}{bV^2}.$$

or, since ψ is always a small angle,

$$\psi = \frac{2\mu}{bV^2}.$$

The transverse velocity imparted by the encounter is

$$V\psi = \frac{2\mu}{bV}.$$

If U is the initial relative velocity of the two stars, σ the distance of nearest approach calculated as if no deflection were to take place,

$$V = \frac{m_2}{m_1 + m_2} U,$$

$$b = \frac{m_2}{m_1 + m_2} \sigma.$$

Also

$$\mu = \gamma m_2 \left(\frac{m_2}{m_1 + m_2}\right)^2,$$

where γ is the constant of gravitation.

Hence the transverse velocity imparted is

$$\frac{2\gamma m_2}{U\sigma}.$$

It may be noted that this expression does not involve m_1, so that the tendency towards equipartition of energy is not indicated in the formula. Equipartition appears to be a third-order effect depending on ψ^3, which has been neglected in the foregoing analysis.

Close approaches which produce an appreciable deflection

ψ are exceedingly rare. Evidently the probability of such an event happening can be calculated when the density of stellar distribution is known. It is of greater importance to determine what is the cumulative effect of the large number of infinitesimal encounters experienced by a star in the course of a long period of time. The following discussion is based on an investigation by J. H. Jeans.[2]

We may set two limits, σ_0 and σ_1. The former is the upper limit of distance for sharp encounters producing considerable deflections; these will be treated as exceptional events to be studied separately. The latter is an arbitrary limit beyond which approaches will not be held to constitute an encounter.

Let ν be the number of stars per unit volume.

The mean free-path (as defined by Maxwell) is

$$\frac{1}{\sqrt{2}\pi\nu\sigma_1^2}.$$

Thus in a length of path L the total number of encounters to be expected is

$$N = 2^{\frac{1}{2}}\pi\nu\sigma_1^2 L.$$

The transverse velocity imparted at any encounter has been shown to be

$$\frac{2\gamma m_2}{U\sigma}.$$

The average value of the relative velocity U is somewhat greater than the velocity v of the star S_1 relative to the stellar system, for collisions will most frequently occur with those stars which are coming to meet S_1.

The average value of $1/\sigma$

$$=\int_0^{\sigma_1}\frac{1}{\sigma}\,2\pi\sigma\,d\sigma \div \int_0^{\sigma_1} 2\pi\sigma\,d\sigma$$
$$=2/\sigma_1.$$

As each encounter takes place in a haphazard direction, the individual contributions of transverse velocity must be compounded according to the theory of errors. Thus the probable resultant of N encounters is proportional

to $\sqrt{N}$; and, generally, the square of the probable resultant will be the sum of the squares of the individual deflections. Thus in averaging different sorts of encounters, we ought to use the mean-square values.

The mean-square value of $\frac{1}{\sigma}$ is somewhat greater than $\frac{2}{\sigma_1}$.* We may conveniently regard this excess as roughly cancelling the excess of U over v and write the resultant transverse velocity after N encounters

$$= \frac{4\gamma m}{v\sigma_1}\sqrt{N}.$$

and the resultant deflection (in radians)

$$= \frac{4\gamma m\sqrt{N}}{\sigma_1 v^2}.$$

Substituting the value of N, the deflection becomes

$$\frac{4\gamma m}{v^2} \times 2^{\frac{1}{2}}\pi^{\frac{1}{2}}\nu^{\frac{1}{2}}L^{\frac{1}{2}}.$$

The following numerical results are deduced by Jeans, on the assumption that the density of stellar distribution is one star to a sphere of radius one parsec, and that the average mass of a star is five times that of the solar

* As the mean-square value appears at first sight to be infinite some further explanation may be desired. The ratio of the mean-square value of $\frac{1}{\sigma}$ to the simple mean is

$$\left\{\int_0^{\sigma_1} 2\pi\sigma d\sigma \, . \, \frac{1}{\sigma^2} \div \pi{\sigma_1}^2\right\}^{\frac{1}{2}} \div \left\{\int_0^{\sigma_1} 2\pi\sigma d\sigma \, . \, \frac{1}{\sigma} \div \pi{\sigma_1}^2\right\}$$
$$= \left\{\tfrac{1}{2}\Big[\log\sigma\Big]_0^{\sigma_1}\right\}^{\frac{1}{2}}$$

which is analytically infinite.

If, however, we reserve sharp encounters for separate consideration we may set σ_0 instead of 0 for the lower limit. For a deflection of 2°, σ_0 will be about 500 astronomical units. σ_1 need certainly not be taken greater than 500,000,000 astronomical units, for an encounter at that distance would last an indefinite time. With these values

$$\sqrt{\left(\tfrac{1}{2}\log\frac{\sigma_1}{\sigma_0}\right)} = 2{\cdot}6.$$

Thus the mean-square value is not more than 2·6 times the simple mean, sudden deflections of over 2° being reserved.

system. This density is somewhat greater than that which we have regarded as probable, and, accordingly, the results may exaggerate a little the disturbance due to encounters.

Jeans finds that for an average star a deflection of 1° may be expected after 3,200 million years In addition to this there is a small "expectation" of deflection by violent encounters. From such a cause a deflection of 2° or more might be expected once in a period of 8×10^{11} years. The meaning of these figures may be illustrated by a definite instance. Considering a moving cluster, all the stars of which have equal and parallel velocities of 40 km. per sec., let a star be considered to continue a member of the main stream so long as its direction of motion does not diverge by more than 2°. After 100 million years only 1 in 8,000 of the original members will be lost by violent encounters, and the remainder will make angles with the main stream of which the average amount is only 10′. After 3,200 million years the loss will be 1 in 250, and the average angle of the remainder will be 1°. After 80,000 million years one-tenth of the original members have been lost by sharp encounters, but the average angle of the remainder is 5°. The ultimate dissolution thus takes place mainly by gradual scattering and not by sharp encounters.

It must not be overlooked that a cluster possesses a certain cohesion of its own, which may resist the minute scattering force to which it is subjected. The cluster is a place where the stars are grouped more densely than in the rest of space and a gravitational force is exerted on those which tend to stray away from it. As we cannot even roughly estimate the whole number of stars in any of the clusters, it is not possible to determine the amount of this force; but a simple calculation will show that it must play some part in keeping the cluster together. Taking Boss's Taurus-stream, which moves at the rate of 40 km. per second, it has been stated that a deflection of 1° may be expected after 3,200 million years, which is equivalent

to 1′ after a million years. This deflection is equivalent to a transverse velocity of 0·012 parsec per million years. It is easily shown that the probable transverse displacement after N million years will be $\dfrac{0{\cdot}012 \times N^{\frac{3}{2}}}{\sqrt{3}}$ parsecs. At the present time the stars of this cluster have spread away from the mean position by an average distance of about 3 parsecs. The corresponding value of N is 57. This calculation gives 57 million years as an upper limit of the age of the Taurus Cluster, assuming that its present extension is wholly due to encounters. The result depends on the rather high value of the stellar density used by Jeans, but with a much smaller density the period is still unreasonably short. It is clear therefore that the dissolving effect of the encounters has been very largely counteracted, and presumably the opposing circumstance is the mutual gravitation of the members of the cluster.

This consideration does not destroy the force of our previous argument. The cohesion of the cluster is only important because the dissolving forces are so excessively minute. Jeans's calculation applies directly to an independent star, and shows that it can pursue its course practically unmolested; whilst the observational evidence is that the effect of encounters is so small that even the minute attraction in a moving cluster is sufficient to counteract it. The evidence, observational and theoretical, seems so conclusive that we have no hesitation in accepting it as the basis of stellar dynamics. The apparent analogy with the kinetic theory of gases is rejected altogether, and it is taken as a fundamental principle that *the stars describe paths under the general attraction of the stellar system without interfering with one another.*

Let us now attempt to estimate the order of magnitude of the general attraction, or the time required for a star to describe its orbit. If the stellar system is not spherically symmetrical the orbits will not in general be closed

paths. But for our estimates we need no precise definition of the periodic time ; we want to know roughly how long it takes for a star to pass from one side to the other of the sidereal system and back again. Within a sphere of uniform density all stars would describe elliptic orbits about the centre isochronously, whatever the initial conditions. The period depends only on the density and is independent of the size of the sphere. The greater the density the less will be the period according to the relation $T \propto \rho^{-\frac{1}{2}}$. In an ellipsoidal system, the component motions along the principal axes will be simple harmonic but with differing periods; the period in a spherical system of the same density will be intermediate between them. Thus the period calculated simply from the density on the assumption of a spherical distribution will give a general idea of the period in the actual universe.

We have estimated the number of stars in a sphere of radius 5 parsecs to be 30 ; as these stars are mostly fainter than the Sun, we shall take the mass in this sphere to be only 10 times the Sun's. If this is an underestimate, and it may be very much underestimated on account of the possible presence of dark stars, the period obtained will be an upper limit. With the adopted density the result is 300,000,000 years. This is less than current estimates of the age of the solid crust of the earth. Thus the Sun and other stars of like maturity must have described at least one and probably many circuits, since they came into being.* We are justified in thinking of the stellar orbits as paths that have actually been traversed, and not as mere theoretical curves.

The problems on which dynamics would be expected to throw some light are numerous. Why have the stars

* We have no means of estimating the age of the stellar system, but perhaps there is no harm in having some such figure as 10^{10} years at the back of our minds in thinking of these questions.

in the early stages very small velocities? Why do these velocities afterwards increase? In particular, how do the stars acquire the velocities at right angles to the original plane of distribution, which cause the latest types to be distributed in a nearly spherical form? How are the two star-streams to be explained? What is the meaning of the third stream, Drift O? Can the partial conforming to Maxwell's law be accounted for? What prevents the collapse of the Milky Way?

Some of these problems seem to be at present quite insoluble. Indeed, it must be admitted that very little progress has been made in the application of dynamics to stellar problems. What has been accomplished is rather of the nature of preparatory work. It has been shown that stellar dynamics is a different study from gas-dynamics, and, indeed, from the theory of any type of system that has yet been investigated. A regular progression may be traced through rigid dynamics, hydrodynamics, gas-dynamics to stellar dynamics. In the first all the particles move in a connected manner; in the second there is continuity between the motions of contiguous particles; in the third the adjacent particles act on one another by collision, so that, although there is no mathematical continuity, a kind of physical continuity remains; in the last the adjacent particles are entirely independent. A new type of dynamical system has therefore to be considered, and it is probably necessary first to work out the results in simple cases and to become familiar with the general properties, before attempting to solve the complex problems which the actual stellar universe presents. This has been the mode of development in the other branches of dynamics.

The natural starting point is to investigate the possible steady states of motion. It will be understood that we are not here referring to the ultimate steady state in which the gas-distribution of velocities prevails, but a state which

remains steady so long as the effect of the encounters is negligible. The actual stellar system may or may not be in such a state; we can best hope to settle the question by working out the consequences of making that hypothesis.

For systems possessing globular symmetry a number of types of steady motion have been found and investigated.[3] None of those discovered up to the present give a reasonable approximation to the actual distribution of velocities; but the failures seem to narrow down very considerably the field in which possible solutions may be obtained. A self-consistent dynamical model, possessing some at least of the main features of stellar motions, would be a most useful and suggestive adjunct in many kinds of investigation, and it would seem to be well worth while continuing the search for suitable systems.

There is one problem relating to the actual stellar system which may be kept in view even at this early stage. H. H. Turner[4] has made an interesting suggestion, which gives a possible explanation of the two star-streams. The problem is to account for preferential motion to and fro in one particular line. Suppose that the stars move in orbits which are in general very elongated, somewhat like the cometary orbits in the solar system. The stellar motions then will be preferentially radial rather than transverse. If the Sun is at a considerable distance from the centre of the stellar system, the line joining that centre to the Sun will be a direction of preferential motion for those stars which are sufficiently near us to have sensible proper motion. Even if the eccentricity of the Sun is not very great, an effect of the character of star-streaming will be observed. We have always assumed that the convergence of the apparent directions of star-streaming in different parts of the sky was evidence that the true directions were parallel; but a convergence of the true directions is an equally possible interpretation. It is quite possible that the preferential motions may be towards

or away from a point at a finite distance rather than parallel to a line. It is difficult to say whether such a hypothesis would prove satisfactory in detail; but at least there is no obvious objection to it.

It may be asked, Why should the stellar orbits be very elongated? The reason that may be assigned is that they start initially with very small velocities. We must suppose that their velocities in later stages are mainly acquired by falling towards the centre of the system, and it is only natural that they should be preferentially radial. Indeed, surprise has sometimes been expressed that there is so little indication of the preponderating radial movements that might be expected. Is it possible that the perplexing phenomenon of the two star-streams is just this indication?

The great difficulty is that, if the motions are mainly radial, it seems inevitable that there should be a great congestion of stars in the neighbourhood of the centre—greater than we care to accept as possible. In the systems that have been worked out up to the present, it has not been possible to obtain sufficient preferential motion without too great a density at the centre; but we cannot as yet conclude that this holds generally. Of course, it is possible to argue that one of the dense patches of stars either in the Cygnus or Sagittarius regions of the Milky Way is the actually congested centre of the stellar system.

A few remarks may be made on the other problems suggested on p. 256. The birth of a star without motion does not seem to present so much difficulty as has sometimes been supposed. It is not necessary to suppose that the primordial matter from which it arises is not subject to gravitation (though there is nothing inherently improbable in such a speculation). Presumably a star is formed by the gathering together of meteoric or gaseous material in some portion of space. Now, we know that, if we were to lump together a thousand stars, their individual motions would

practically cancel and the resultant super-star would be nearly at rest. Similarly, in forming a single star, the individual motions produced by gravitation in the materials of which it is composed might cancel so that the star would start from rest. It is interesting to note that by this process the average initial velocity of a star composed of N constituents would vary *ceteris paribus* as $N^{-\frac{1}{2}}$, the velocities of the constituents being assumed to be haphazard. This might appear to lead to an equipartition of energy between stars of different masses, the mass being proportional to N. But unless the number of constituents is very small their velocities would have to be enormous, and the suggestion does not seem to be tenable. Moreover, it is difficult to see how a moving cluster could be formed.*

An increase of velocity in the next stage is the natural result of the central attraction of the stellar system. If the star starts existence with no motion, it must start from apcentron, and at all other points of the orbit its velocity would be greater. After the star is old enough to have described one quadrant of its orbit, we cannot look for any increase in velocity from this cause; in other words, the progressive increase of velocity should cease after the first 100,000,000 years. But we can scarcely compress the development from Type B to Type M into that period. Another difficulty is that the motion produced by the central attraction would be mainly in the plane of the galaxy; there is no explanation of the motions perpendicular to that plane acquired by the stars of the later types. It does not seem possible to account for these extra-galactic motions in any simple way.

I see no alternative to supposing that the K and M stars have been formed originally in a more globular distribution than the early type stars. It may be that the

* This suggestion (with the objections to it) was mentioned to me by Prof. A. Schuster.

birth of stars was for some reason retarded in the galactic plane, and that this is the reason why early type stars abound there. Perhaps a more likely supposition is that massive stars with slow development have formed where the material was rich, and small stars with rapid development have formed where the star-stuff was scanty. Thus the outlying parts of the stellar system away from the galaxy have given birth to the small stars which have rapidly reached the M stage; and these, having fallen in from a great distance, have acquired large velocities. The regions of the galactic plane, richly supplied with the necessary material, have formed large stars, which are only slowly developing, and these have remained moving in the galactic plane. The hypothesis in outline seems fairly plausible; but so long as the difficulty of the double character of the Type M stars remains we cannot regard any explanation as complete.

The foregoing suggestion is also applicable if we adopt Russell's hypothesis (p. 170). His view is that only the most massive stars are able to heat themselves up to the high temperatures characteristic of Types B and A. Accordingly, these types will only have originated where the star-forming material was rich, and their galactic concentration and small velocities can be accounted for.

The problem of the equilibrium of the Milky Way is another subject for reflection. It seems necessary to grant that it is in some sort of equilibrium; that is to say, the individual stars do not oscillate to and fro across the stellar system in a 300,000,000 year period, but remain concentrated in the clusters, which they now form. The only possible explanation in terms of known forces is that the Milky Way as a whole is in slow rotation, a condition which has been considered by H. Poincaré.[1] To obtain some notion of the order of magnitude of the rotation, let us assume the total mass of the inner stellar system to be

10^9 times the Sun's mass, and the distance of the Milky Way to be 2,000 parsecs; the angular velocity for equilibrium will then be $0''\cdot 5$ per century. It may be pointed out that Charlier[5] has found that the node of the invariable plane of the solar system on the plane of the Milky Way has a direct motion amounting to $0''\cdot 35$ per century, a motion which might equally well be expressed as a rotation of the stars in the plane of the Milky Way in a retrograde direction. Perhaps it would be straining the result too far to regard this as evidencing the truth of our surmise.

With this brief reference to the dynamical aspect of the problem, we conclude our survey of the structure of the stellar system. The results discussed have, with few exceptions, come to light during the last fifteen years; but they are the outcome of a century's labour of preparation. The proper motions now used are based on observations which go back to the time of Bradley; and the modern instrumental methods, which are now yielding parallaxes and proper motions for discussion, have a long history of gradual development behind them. The progress of stellar investigation must not be measured by the few conclusions to which we have been able to give definite statement. In the future the fruits of these labours will be reached far more fully.

Meanwhile the knowledge that has been attained shows only the more plainly how much there is to learn. The perplexities of to-day foreshadow the discoveries of the future. If we have still to leave the stellar universe a region of hidden mystery, yet it seems as though, in our exploration, we have been able to glimpse the outline of some vast combination which unites even the farthest stars into an organised system.

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